Gauge theory of elementary particle physics

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## Problems and solutions

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# Preface

Students of particle physics often find it difficult to locate resources to learn calculational techniques. Intermediate steps are not usually given in the research literature. To a certain extent, this is also the case even in some of the textbooks. In this book of worked problems we have made an effort to provide enough details so that a student starting in the field will understand the solution in each case. Our hope is that with this step-by-step guidance, students (after first attempting the solution themselves) can develop their skill, and confidence in their ability, to work out particle theory problems.

This collection of problems has evolved from the supplemental material developed for a graduate course that one of us ( L.F.L.) has taught over the years at Carnegie Mellon University, and is meant to be a companion volume to our textbook *Gauge Theory of Elementary Particle Physics* (referred to as CL throughout this book) rather than a complete assemblage of gauge theory problems. Nevertheless, it has a self-contained format so that even a reader not familiar with CL can use it effectively. All the problems (usually with several parts) have been given a descriptive title. By simply inspecting the table of contents readers should be able to pick out the areas they wish to tackle.

Several new subjects have entered in the field in the fifteen years since the original writing of CL. Although we have not revised the book to incorporate them because we would not be able to do them justice, we hope this set of problem/ solution presentations is the first step towards remedying the situation. We have incorporated a number of new topics and developed further those that were only introduced briefly in the original text. Listed below are some of these areas:

- Relations among different renormalization schemes
- Further applications of the path-integral formalism
- General relativity as a gauge theory
- Superconductivity as a Higgs phenomenon
- Non-linear sigma model and chiral symmetry
- Path integral derivation of the axial anomaly
- Infrared and collinear divergence in QCD
- Further examples of the parton model phenomenology
- QCD and $\Delta I = \frac{1}{2}$ rule in the non-leptonic weak decays
- More on gauge theories of lepton number violation
- Group theory of grand unification
- Further examples of solitons

Many people have helped us in preparing this book. Our thanks go particularly to all the students who have taken the course and have worked through a good part of these problems. One of us (T.P.C.) also wishes to acknowledge the enjoyable

hospitality of the Santa Cruz Institute of Particle Physics when finishing up this project . The original literature has only been referenced casually, and we apologize to the authors whose work we may have neglected to cite.

This book and CL share a page on the World Wide Web at the URL `http://www.umsl.edu/~tpcheng/gaugebooks.html`. Misprints or other corrections brought to our attention will be posted on this page. We would be grateful for any comments about these books.

*St. Louis* T.P.C.
*Pittsburgh* L.F.L.
January 1999

# Contents

# 1 Field quantization

## 1.1 Simple exercises in $\lambda\phi^4$ theory

In $\lambda\phi^4$ theory, the interaction is given by

$$H_I = \frac{\lambda}{4!}\phi^4(x). \tag{1.1}$$

**(a)** Show that, to the lowest order in $\lambda$, the differential cross-section for two-particle elastic scattering in the centre-of-mass frame is given by

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{128\pi^2 s} \tag{1.2}$$

where $s = (p_1 + p_2)^2$, with $p_1$ and $p_2$ being the momenta of the incoming particles.

**(b)** Use Wick's theorem to show that the graphs in Fig. 1.1 have the symmetry factors as given. Also, check that these results agree with a compact expression for the symmetry factor

$$S = g \prod_{n=2,3,\ldots} 2^\beta (n!)^{\alpha_n} \tag{1.3}$$

where $g$ is the number of possible permutations of vertices which leave unchanged the diagram with fixed external lines, $\alpha_n$ is the number of vertex pairs connected by $n$ identical lines, and $\beta$ is the number of lines connecting a vertex with itself.

**(c)** Show that the two-point Green's function satisfies the relation

$$(\Box_x + \mu^2)\langle 0|T(\phi(x)\phi(y))|0\rangle = \frac{\lambda}{3!}\langle 0|T(\phi^3(x)\phi(y))|0\rangle - i\delta^4(x-y).$$

Also verify this relation diagrammatically to first order in $\lambda$.

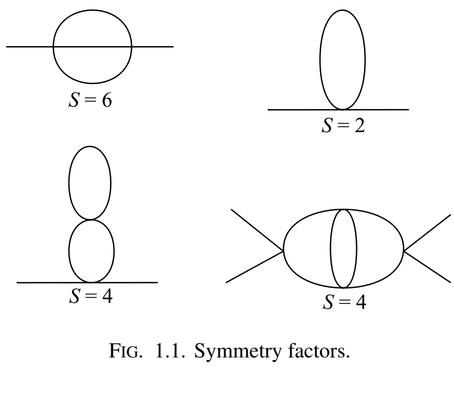



FIG. 1.1. Symmetry factors.

**(d)** A Green's function involving the composite operator $\Omega(x) = \frac{1}{2}\phi^2(x)$ is defined as

$$G_\Omega^{(n)}(x;\ x_1, \ldots, x_n) = \langle 0|T(\Omega(x)\phi(x_1)\cdots\phi(x_n))|0\rangle. \tag{1.4}$$

Write down, to the first order in $\lambda$, the various contributions to $G_\Omega^{(2)}(x;\ x_1, x_2)$.

### Solution to Problem 1.1

**(a)** The tree diagram for a two-particle elastic scattering is shown in Fig. 1.2. Thus to this order the scattering amplitude is simply $T = -i\lambda$ giving rise to a differential cross-section: (see CL-Appendix A for rules of cross-section calculation):

$$\begin{aligned} d\sigma = {} & \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\frac{1}{2E_1}\frac{1}{2E_2}|-i\lambda|^2 \frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \\ & \times (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\tfrac{1}{2}. \end{aligned} \tag{1.5}$$

The last $\frac{1}{2}$ factor is inserted to account for the presence of two identical particles in the final state.

We then have the phase space factor of

$$\rho = \int (2\pi)^4\delta^4(\bar{p} - p_3 - p_4)\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \tag{1.6}$$

where $\bar{p} = p_1 + p_2$. In the centre-of-mass frame, the four momenta can be parametrized as $p_1 = (E, \mathbf{p})$, $p_2 = (E, -\mathbf{p})$, $p_3 = (E', \mathbf{p}')$, and $p_4 = (E', -\mathbf{p}')$. After performing the $d^3\mathbf{p}_4$ integration, the phase factor becomes

$$\begin{aligned} \rho &= \int (2\pi)^{-2}\delta(2E - 2E')\frac{d^3\mathbf{p}'}{4E'^2} \\ &= \int (2\pi)^{-2}\delta(2E - 2E')\frac{p'E'\,dE'}{4E'^2}\,d\Omega \\ &= \frac{|\mathbf{p}|}{32\pi^2 E}\,d\Omega \end{aligned} \tag{1.7}$$

and thus the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\frac{1}{4E^2}\frac{|\mathbf{p}|}{32\pi^2 E}\frac{\lambda^2}{2}. \tag{1.8}$$

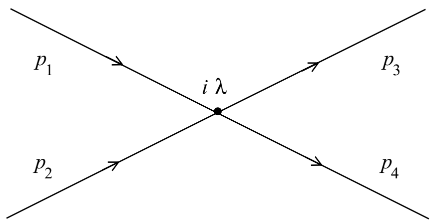



FIG. 1.2.

After substituting the flux of $|\mathbf{v}_1 - \mathbf{v}_2| = |(\mathbf{p}_1/E_1) - (\mathbf{p}_2/E_2)| = 2|\mathbf{p}|/E$ and the invariant variable $s = (p_1 + p_2)^2 = 4E^2$ into the above expression we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{128\pi^2 s}. \tag{1.9}$$

**(b) (i)** The diagram in Fig. 1.3 corresponds to a second-order term in the perturbation expansion

$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \int d^4y_1\, d^4y_2\, \langle 0|T[\phi(x_1)\phi(x_2){:}\phi(y_1)\phi(y_1)\phi(y_1)\phi(y_1){:}\, {:}\phi(y_2)\phi(y_2)\phi(y_2)\phi(y_2){:}]|0\rangle. \tag{1.10}$$

The amplitude like the one above but with the interchange $y_1 \leftrightarrow y_2$ has the same contribution. This doubling cancels the first factor of $\frac{1}{2!}$ in the above expression, which comes from the Taylor expansion.

Wick's expansion leads to the following contractions. There are four ways to contract $\phi(x_1)$ with any one of the $\phi(y_1)$s and similarly four ways to contract $\phi(x_2)$ with any one of the $\phi(y_2)$s; then there are 3! ways to contract the remaining pairs of $\phi(y_1)$ and $\phi(y_2)$. The (inverse) symmetry factor is

$$S^{-1} = \left(\frac{1}{2!}\cdot 2\right)\left(\frac{1}{4!}\right)^2 4\cdot 4\cdot 3! = \frac{1}{3!}. \tag{1.11}$$

This checks with the result obtained by using eqn (1.3) directly, because $g = 1$, $\alpha_3 = 1$, and $\beta = 0$.

**(ii)** The diagram in Fig. 1.4 is first order in the coupling

$$\frac{-i\lambda}{4!}\int d^4y\, \langle 0|T[\phi(x_1)\phi(x_2){:}\phi(y)\phi(y)\phi(y)\phi(y){:}]|0\rangle. \tag{1.12}$$

There are four ways to contract $\phi(x_1)$ with any one of the $\phi(y)$s and three ways to contract $\phi(x_2)$ with any one of the remaining three $\phi(y)$s. Hence

$$S^{-1} = \frac{4\cdot 3}{4!} = \frac{1}{2}. \tag{1.13}$$

This checks with the result obtained from eqn (1.3), since $g = 1$, $\alpha_n = 0$, and $\beta = 1$.

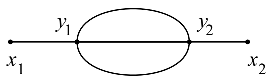



FIG. 1.3.

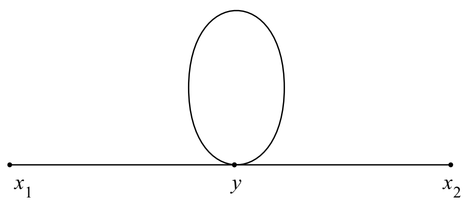



FIG. 1.4.

**(iii)** The diagram in Fig. 1.5, like the one in Fig. 1.2, corresponds to the second-order term as given in (i).

The multiplicity is determined by noting that there are four ways to contract $\phi(x_1)$ with any one of the $\phi(y_1)$s and three ways to contract $\phi(x_2)$ with any one of the remaining three $\phi(y_1)$s. And there are $\binom{4}{2} = 4 \cdot 3$ ways to contract the remaining $\phi(y_1)$ pair to all the possible pairs out of the four $\phi(y_2)$s.

$$S^{-1} = \left(\frac{1}{2!} \cdot 2\right) \frac{1}{(4!)^2} \cdot 4 \cdot 3 \cdot 4 \cdot 3 = \frac{1}{4}. \tag{1.14}$$

Equation (1.3) also yields $S = 4$ because in this case $g = 1$, $\alpha_2 = 1$, and $\beta = 1$.

**(iv)** Figure 1.6(a) is a fourth-order diagram. There are $\frac{4!}{2}$ such diagrams corresponding to 4! ways to permute the $y_{1,2,3,4}$ positions for a fixed $x_{1,2,3,4}$, and the two categories of diagrams as illustrated in Fig. 1.6(b) are actually identical. Thus the Taylor series factor of $\frac{1}{4!}$ is only partially compensated.

For brevity, for the remaining part of the amplitude we will only display the position factors of the fields

$$x_1x_2x_3x_4 \quad y_1y_1y_1y_1 \quad y_2y_2y_2y_2 \quad y_3y_3y_3y_3 \quad y_4y_4y_4y_4 \tag{1.15}$$

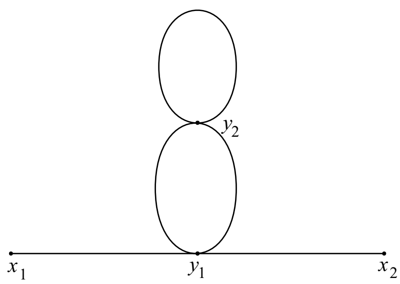



FIG. 1.5.

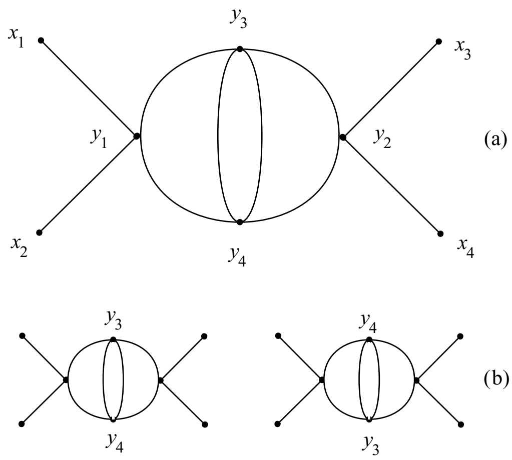



FIG. 1.6.

and examine its combinatorics. There are $4 \cdot 3$ ways to contract $\phi(x_1)$ and $\phi(x_2)$ with the four $\phi(y_1)$s, and the same number of ways between $\phi(x_3)$ and $\phi(x_4)$ and the $\phi(y_2)$s. For the remaining two $\phi(y_1)$s to contract into the respective four $\phi(y_3)$s and $\phi(y_4)$s there are $2 \cdot 4 \cdot 4$ ways. Similarly, for the remaining two $\phi(y_1)$s to contract into the respective remaining three $\phi(y_3)$s and $\phi(y_4)$s there are $2 \cdot 3 \cdot 3$ ways. Finally, there are two ways for the remaining two $\phi(y_3)$s and $\phi(y_4)$s to contract into each other.

$$\begin{aligned} S^{-1} &= \left(\frac{1}{4!} \cdot \frac{4!}{2}\right) \frac{1}{(4!)^4} \cdot (4 \cdot 3)^2 \cdot (2 \cdot 4 \cdot 4) \cdot (2 \cdot 3 \cdot 3) \cdot 2 \\ &= \frac{1}{2} \frac{(4 \cdot 3)^4 \cdot 2^3}{(4!)^4} = \frac{1}{4}. \end{aligned} \tag{1.16}$$

This again checks with eqn (1.3), since $g = 2$, $\alpha_2 = 1$, and $\beta = 0$.

**(c)** First we show that the differentiation of the two-point function with respect to $x$ yields

$$\begin{aligned} \partial_x^\mu \langle 0|T(\phi(x)\phi(y))|0\rangle &= \langle 0|T(\partial^\mu \phi(x)\phi(y))|0\rangle \\ &\quad + \langle 0|[\phi(x), \phi(y)]|0\rangle \delta(x_0 - y_0) \end{aligned} \tag{1.17}$$

where the equal-time commutator actually vanishes. Differentiating for the second time we have

$$\begin{aligned} \Box_x \langle 0|T\phi(x)\phi(y)|0\rangle &= \langle 0|T(\Box\phi(x)\phi(y))|0\rangle \\ &\quad + \langle 0|[\partial_0\phi(x), \phi(y)]|0\rangle \delta(x_0 - y_0). \end{aligned} \tag{1.18}$$

From the equation of motion $\Box\phi(x) = -\mu^2\phi - (\lambda/3!)\phi^3$ and the canonical commutation relation $[\partial_0\phi(x), \phi(y)]\,\delta(x_0 - y_0) = -i\delta^4(x - y)$, we then obtain the result stated in the problem.

$$(\Box_x + \mu^2)\langle 0|T(\phi(x)\phi(y))|0\rangle = -\frac{\lambda}{3!}\langle 0|T(\phi^3(x)\phi(y))|0\rangle - i\delta^4(x - y). \tag{1.19}$$

To verify this relation diagrammatically we note that the first order in $\lambda$ diagrams for the two-point function are given in Fig. 1.7(a).

The Feynman diagrams lead us to the relation

$$\begin{aligned} \langle 0|T(\phi(x)\phi(y))|0\rangle &= i\Delta_F(x - y) + \left(\frac{-i\lambda}{2}\right) \\ &\quad \times \int d^4z\,[i\Delta_F(x - z)][i\Delta_F(z - y)]i\Delta_F(0) \end{aligned} \tag{1.20}$$

Using the relation $(\Box_x + \mu^2)\Delta_F(x - y) = -\delta^4(x - y)$, we obtain the left-hand side of eqn (1.19) to first order:

$$\begin{aligned} &-i\delta^4(x - y) + \left(\frac{-i\lambda}{2}\right) \int d^4z\,[i\delta^4(x - z)][i\Delta_F(z - y)]i\Delta_F(0) \\ &= -\frac{\lambda}{2}\Delta_F(x - y)\Delta_F(0) - i\delta^4(x - y). \end{aligned} \tag{1.21}$$

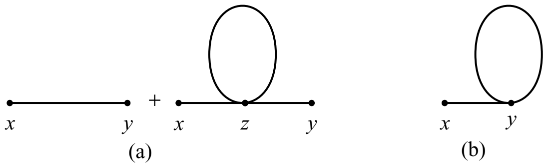



FIG. 1.7.

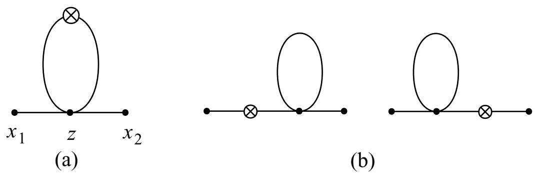



FIG. 1.8.

Writing out the Green's function on the right-hand side, we have

$$-\frac{\lambda}{3!}\langle 0|T(\phi(x)\phi(x)\phi(x)\phi(y))|0\rangle = -\frac{\lambda}{2} i\Delta_F(x-y) i\Delta_F(0). \tag{1.22}$$

Equations (1.21) and (1.22) clearly show that the relation (1.19) is satisfied. The Feynman diagram for eqn (1.22) is shown in Fig. 1.7(b).

**(d)** There are three first-order diagrams for the two-point function

$$G_\Omega^{(2)}(x;\ x_1, x_2) = \langle 0|T(\tfrac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2))|0\rangle. \tag{1.23}$$

We shall explicitly work out the case of diagram (a) in Fig. 1.8.

$$\begin{aligned} &\langle 0|T\tfrac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)\left(\frac{-i\lambda}{4!}\right)\int d^4y\phi^4(y)|0\rangle \\ &= \left(\frac{-i\lambda}{2}\right)\int d^4y[i\Delta_F(x_1-y)][i\Delta_F(x_2-y)][i\Delta_F(x-y)]^2. \end{aligned} \tag{1.24}$$

The symmetry factor of $S = 2$ can be understood by noting that there are $4\cdot 3$ ways to contract between $\phi(x_1)\phi(x_2)$ and two $\phi$s in $\phi^4(y)$, and 2 ways to contract $\phi^2(x)$ with the remaining two $\phi(y)$s. Thus the 4! factor in the coupling is cancelled, and we are left with the $\frac{1}{2}$ factor from the composite operator. The diagrams in Fig. 1.8(b) can be worked out in the same way. Their symmetry factors are also 2.

## 1.2 Auxiliary field

The Lagrangian density for a set of real scalar field $\phi^a$, $a = 1, 2, \ldots, N$, is given as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) - \frac{\mu^2}{2}\phi^a\phi^a - \frac{\lambda}{8}(\phi^a\phi^a)^2. \tag{1.25}$$

**(a)** Work out the basic vertices in this theory by calculating the four-point amputated Green's function to the first order in $\lambda$.

**(b)** Consider the Lagrangian density

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \frac{\mu^2}{2}\phi^a\phi^a + \frac{1}{2\lambda}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a \tag{1.26}$$

where $\sigma(x)$ is another scalar field.

**(i)** Show that if we eliminate $\sigma(x)$ by using the equation of motion, we end up with the Lagrangian in eqn (1.25).

**(ii)** If we do not eliminate $\sigma(x)$, and take the propagator for $\sigma(x)$ in the momentum space to be $-i\lambda$ (which can be justified by adding a term $(\epsilon/2)(\partial_\mu\sigma)(\partial^\mu\sigma)$ and then the limit of $\epsilon \to 0$ after the propagator has been worked out), show that $\mathcal{L}'$ gives the same basic vertex for $\phi(x)$ as that given in part (a).

### Solution to Problem 1.2

**(a)** To the first order in $\lambda$, the four-point Green's function with the four external lines carrying the internal indices $a, b, c, d$ is given as

$$\langle 0|T\phi^a\phi^b\phi^c\phi^d \left(\frac{-i\lambda}{8}\right)\phi^i\phi^i\phi^j\phi^j|0\rangle \tag{1.27}$$

where we have grouped the four fields in the interaction term into two pairs labelled $i$ and $j$, respectively—repeated indices are always summed over.

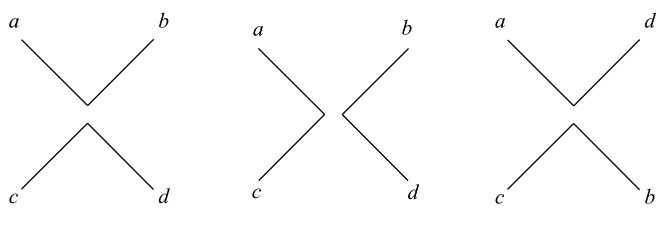
As displayed in Fig. 1.9, there are two ways $\phi^i$s can be contracted with $\phi^a\phi^b$, and two ways between $\phi^j$s and $\phi^c\phi^d$; these four ways are to be multiplied by 2 corresponding to the interchange $i \leftrightarrow j$. Thus the factor of 8 is cancelled and the vertex is given by $-i\lambda\delta^{ab}\delta^{cd}$. There are of course other ways we can pair off the four external lines. Removing the propagators for the external lines, we have the basic vertex for this theory:

$$-i\lambda(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}). \tag{1.28}$$

**(b) (i)** Since the $\mathcal{L}'$ does not contain the $\partial_\mu\sigma$ field, the equation of motion for the $\sigma$ field $\partial\mathcal{L}'/\partial\sigma = 0$ is simply a constraint equation: $\sigma/\lambda = \frac{1}{2}\phi^a\phi^a$. Substituting this condition into the $\mathcal{L}'$ Lagrangian density, the $\sigma$-dependent part becomes

$$\frac{1}{2\lambda}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a = \frac{\lambda}{8}(\phi^a\phi^a)^2 - \frac{\lambda}{4}(\phi^a\phi^a)^2 = -\frac{\lambda}{8}(\phi^a\phi^a)^2 \tag{1.29}$$

and thus $\mathcal{L}' = \mathcal{L}$.




(a) (b) (c)

FIG. 1.9.

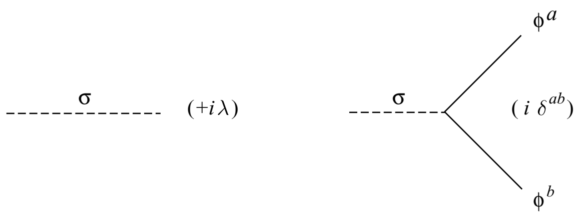



FIG. 1.10. Feynman rules for the Lagrangian $\mathcal{L}'$.

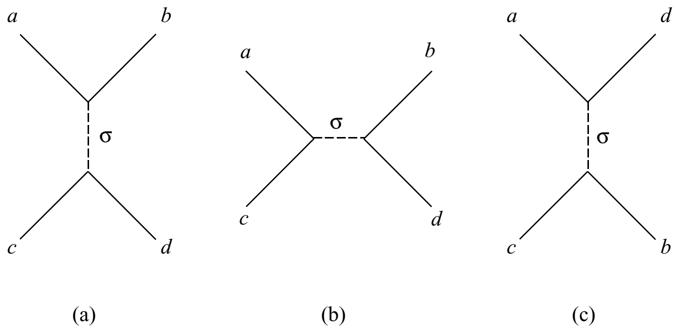



FIG. 1.11.

**(ii)** The Feynman rule from the $\mathcal{L}'$ theory is shown in Fig. 1.10.
From this we can construct the Feynman diagrams for the four-point function in Fig. 1.11.

Diagram (a) yields $(-i\delta^{ab})(+i)(-i\delta^{cd}) = -i\delta^{ab}\delta^{cd}$. Similarly, diagrams (b) and (c) give $-i\delta^{ac}\delta^{bd}$ and $-i\delta^{ad}\delta^{bc}$, respectively.

**Remark.** Very often this kind of auxiliary field is introduced to make the calculation more tractable. For the case here, the use of the $\sigma$-field makes the flow of the internal symmetry indices easier to monitor.

## 1.3 Disconnected diagrams

Consider the unperturbed and perturbative parts of the scalar field theory

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{\mu^2}{2}\phi^2, \quad \mathcal{L}_1 = -\frac{m^2}{2}\phi^2. \tag{1.30}$$

In the perturbation theory, the two-point Green's function is given by

$$\begin{aligned} G^{(2)}(x_1, x_2) &= \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle \\ &= \frac{\langle 0|T(\phi_0(x_1)\phi_0(x_2)\exp[-i\int H'(x)\,dx])|0\rangle}{\langle 0|T(\exp[-i\int H'(x)\,dx])|0\rangle}. \end{aligned} \tag{1.31}$$

Use Wick's theorem to demonstrate explicitly that the respective disconnected graphs in the numerator and denominator cancel.

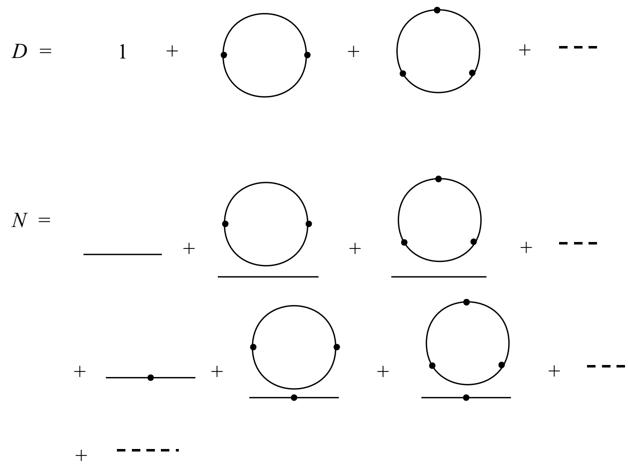



FIG. 1.12.

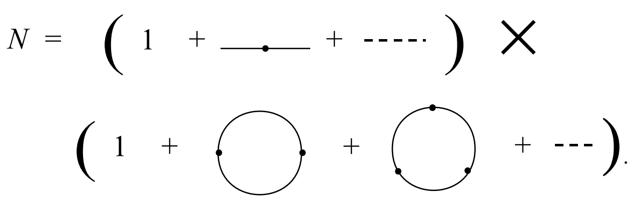



FIG. 1.13.

**Solution to Problem 1.3**

The two-point Green's function

$$G^{(2)}(x_1, x_2) = \frac{\langle 0|T(\phi_0(x_1)\phi_0(x_2)\exp[-i\int H'(x)\,dx])|0\rangle}{\langle 0|T(\exp[-i\int H'(x)\,dx])|0\rangle} \equiv \frac{N}{D} \tag{1.32}$$

has the following Wick's (diagrammatic) expansions, shown in Fig. 1.12, for the denominator $D$ and the numerator $N$, respectively, where the dot represents the 'vertex' of $\mathcal{L}_1 = -(m^2/2){:}\phi^2(x){:}$. This is equivalent to the expansion as shown in Fig. 1.13.

We see that the disconnected contribution has been cancelled.

## 1.4 Simple external field problem

Suppose the Lagrangian for a scalar field is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)^2 - \frac{\mu^2}{2}\phi^2 - J(x)\phi(x) \tag{1.33}$$

where $J(x)$ is a real c-number function.

**(a)** Calculate $\langle 0|\phi(x)|0\rangle$ and the two-point function $\langle 0|T(\phi(x)\phi(0))|0\rangle$ exactly.

**(b)** Treat the term $J(x)\phi(x)$ as a perturbation and calculate $\langle 0|\phi(x)|0\rangle$ and $\langle 0|T(\phi(x)\phi(0))|0\rangle$ to the lowest order in $J(x)$.

### Solution to Problem 1.4

**(a)** The Lagrangian yields the equation of motion

$$(\Box + \mu^2)\phi(x) = -J(x). \tag{1.34}$$

If we define the usual Green's (propagator) function,

$$(\Box + \mu^2)\Delta_F(x-y) = -\delta^4(x-y), \tag{1.35}$$

the field operator can then be written as $\phi(x) = \phi_0(x) + \hat{\phi}(x)$, where $\phi_0(x)$ is a c-number function:

$$\phi_0(x) = \int d^4y\, \Delta_F(x-y)J(y). \tag{1.36}$$

$\hat{\phi}(x)$ satisfies the homogeneous Klein–Gordon equation, $(\Box + \mu^2)\hat{\phi}(x) = 0$, and can be expanded in terms of the usual creation and annihilation operators, satisfying the commutation relation $\left[a(k), a^\dagger(k')\right] = \delta^3(k-k')$:

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} \left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right]. \tag{1.37}$$

Because $\hat{\phi}(x)|0\rangle = \langle 0|\hat{\phi}(x) = 0$ the vacuum expectation value of the unshifted field operator is non-vanishing:

$$\langle 0|\phi(x)|0\rangle = \phi_0(x) = \int d^4y\, \Delta_F(x-y)J(y), \tag{1.38}$$

and the two-point Green's function is also shifted as

$$\begin{aligned}\langle 0|T\phi(x)\phi(0)|0\rangle &= \langle 0|T\phi_0(x)\phi_0(0)|0\rangle + \langle 0|T\hat{\phi}(x)\hat{\phi}(0)|0\rangle \\ &= \phi_0(x)\phi_0(0) + i\Delta_F(x).\end{aligned} \tag{1.39}$$

**(b)** The 'interaction vertex' in the Feynman's diagrams for this theory is given in Fig. 1.14(a).

**(i)** The perturbative expansion for the vacuum expectation value can be represented by a diagram similar to Fig. 1.14(a).

$$\begin{aligned}\langle 0|\phi(x)|0\rangle &= \langle 0|T\phi_I(x)\sum_n \frac{(-i)^n}{n!} \\ &\quad \times \int d^4y_1\, J(y_1)\phi_I(y_1)\cdots d^4y_n J(y_n)\phi_I(y_n)|0\rangle_c \\ &= -i\int d^4y\, \langle 0|T\phi_I(x)\phi_I(y)|0\rangle J(y) \\ &= \int d^4y\, \Delta_F(x-y)J(y) = \phi_0(x).\end{aligned}$$

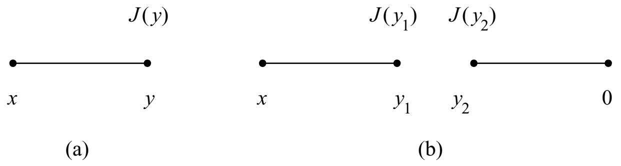



FIG. 1.14.

**(ii)** The perturbative expansion for the two-point Green's function is given by Fig. 1.14(b).

$$
\begin{aligned}
\langle 0|T\phi(x)\phi(0)|0\rangle &= \langle 0|T\phi_I(x)\phi_I(0) \\
&\quad \times \frac{(-i)^2}{2!}\int d^4y_1\, d^4y_2\, J(y_1)J(y_2)\phi_I(y_1)\phi_I(y_2)|0\rangle_c \\
&= \frac{(-i)^2}{2!}\int d^4y_1\, d^4y_2\, [i\Delta_F(x-y_1)i\Delta_F(-y_2)J(y_1)J(y_2) \\
&\quad + (y_1 \leftrightarrow y_2)] \\
&= \phi_0(x)\phi_0(0).
\end{aligned} \tag{1.40}
$$

## 1.5 Path integral for a free particle

Show that the transition amplitude for a free particle (mass $m$) moving in one dimension has the expression

$$
\langle q', t'|q, t\rangle = \left[\frac{m}{2\pi i(t'-t)}\right]^{1/2} \exp\left[\frac{im}{2}\frac{(q'-q)^2}{t'-t}\right]. \tag{1.41}
$$

You should check that this result can be obtained by starting either from the Hamiltonian or the path integral (Lagrangian) representations of the transition amplitude:

$$
\langle q', t'|q, t\rangle = \begin{cases} \langle q'|\exp[-iH(t'-t)]|q\rangle \\ N\int[dq]\exp\left[i\int_t^{t'} dt'' L\right] \end{cases} \tag{1.42}
$$

where $H = p^2/2m$ and $L = (m/2)\dot{q}^2$ and the integration measure in the path integral representation is given by

$$
N[dq] = \lim_{n\to\infty}\left(\frac{m}{2\pi i\Delta}\right)^{n/2}\prod_{i=1}^{n-1} dq_i \tag{1.43}
$$

with $(t'-t)$ being divided into $n$ equal segments of $\Delta$: $t, t_1, t_2, \ldots, t_{n-1}, t' \equiv t_n$, having the corresponding positions $q, q_1, q_2, \ldots, q_{n-1}, q' \equiv q_n$.

**Solution to Problem 1.5**

**(a) The Hamiltonian method**

$$\begin{aligned}\langle q', t'|q, t\rangle &= \langle q'|\exp[-iH(t'-t)]|q\rangle \\ &= \langle q'|\exp\left[\frac{-ip^2}{2m}(t'-t)\right]|q\rangle. \end{aligned} \tag{1.44}$$

Inserting a complete set of momentum states:

$$\begin{aligned}\langle q', t'|q, t\rangle &= \int \frac{dp}{2\pi}\langle q'|\exp\left[\frac{-ip^2}{2m}(t'-t)\right]|p\rangle\langle p|q\rangle \\ &= \int \frac{dp}{2\pi}\exp\left[\frac{-ip^2}{2m}(t'-t) + ip(q'-q)\right]\end{aligned} \tag{1.45}$$

which can be integrated by using the Gaussian integral formula:

$$\int_{-\infty}^{\infty} dx \exp(-ax^2 + bx) = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right). \tag{1.46}$$

In our case, we have $a = (i/2m)(t'-t)$ and $b = i(q'-q)$. Thus,

$$\langle q', t'|q, t\rangle = \left[\frac{m}{2\pi i(t'-t)}\right]^{1/2}\exp\left[\frac{im}{2}\frac{(q'-q)^2}{t'-t}\right]. \tag{1.47}$$

**(b) The path integral method**

The *action* can be written in terms of the space-time intervals as

$$\begin{aligned}S &= \int_t^{t'} L\,dt'' = \int_t^{t'} \frac{m}{2}\dot{q}^2\,dt'' = \frac{m}{2}\sum_{i=1}^{n-1}\left(\frac{q_i - q_{i+1}}{\Delta}\right)^2\Delta \\ &= \frac{m}{2\Delta}\left[(q-q_1)^2 + (q_1-q_2)^2 + \cdots + (q_{n-1}-q')^2\right].\end{aligned} \tag{1.48}$$

Using this and the given integration measure, the transition amplitude can be expressed as

$$\begin{aligned}\langle q', t'|q, t\rangle = \left(\frac{m}{2\pi i}\right)^{n/2}\int\prod_{i=1}^{n-1} dq_i \exp\Bigg\{\frac{im}{2\Delta}\Big[(q-q_1)^2 \\ + (q_1-q_2)^2 + \cdots + (q_{n-1}-q')^2\Big]\Bigg\}.\end{aligned} \tag{1.49}$$

The successive integrals can be calculated by using the formulae for Gaussian integrals of the form

$$\int_{-\infty}^{\infty} dx \exp\left[a(x-x_1)^2 + b(x-x_2)^2\right] = \left[\frac{-\pi}{a+b}\right]^{1/2}\exp\left[\frac{ab}{a+b}(x_1-x_2)^2\right]$$

so that one has

$$\int dq_1 \exp\left\{\frac{im}{2\Delta}\left[(q-q_1)^2+(q_1-q_2)^2\right]\right\}$$

$$=\left[\frac{2\pi i\Delta}{m}\cdot\frac{1}{2}\right]^{1/2}\exp\left[\frac{im}{2\Delta}\frac{(q-q_2)^2}{2}\right]$$

$$\int dq_2 \exp\left\{\frac{im}{2\Delta}\left[\frac{(q-q_2)^2}{2}+(q_2-q_3)^2\right]\right\}$$

$$=\left[\frac{2\pi i\Delta}{m}\cdot\frac{2}{3}\right]^{1/2}\exp\left[\frac{im}{2\Delta}\frac{(q-q_3)^2}{3}\right]$$

$$\int dq_3 \exp\left\{\frac{im}{2\Delta}\left[\frac{(q-q_3)^2}{3}+(q_3-q_4)^2\right]\right\}$$

$$=\left[\frac{2\pi i\Delta}{m}\cdot\frac{3}{4}\right]^{1/2}\exp\left[\frac{im}{2\Delta}\frac{(q-q_4)^2}{4}\right]$$

and so on. In this way, one obtains

$$\begin{aligned}\langle q',t'|q,t\rangle &= \lim_{n\to\infty}\left(\frac{m}{2\pi i\Delta}\right)^{n/2}\left(\frac{2\pi i\Delta}{m}\right)^{(n-1)/2}\left(\frac{1}{2}\frac{2}{3}\cdots\frac{n-1}{n}\right)^{1/2}\\ &\quad\times\exp\left[\frac{im}{2n\Delta}(q-q')^2\right]\\ &=\lim_{n\to\infty}\left(\frac{m}{2\pi i n\Delta}\right)^{1/2}\exp\left[\frac{im(q'-q)^2}{2(t'-t)}\right]\\ &=\left(\frac{m}{2\pi i(t'-t)}\right)^{1/2}\exp\left[\frac{im(q'-q)^2}{2(t'-t)}\right]\end{aligned}\tag{1.50}$$

where one has used $n\Delta=(t'-t)$. This result agrees with that obtained in (i) by using the Hamiltonian as the generator of time evolution.

## 1.6 Path integral for a general quadratic action

We will study the case of the action containing at most quadratic terms

$$S[q]=\int dt\left[a(t)\dot{q}^2+b(t)\dot{q}+c(t)q\dot{q}+d(t)q+e(t)q^2+f(t)\right].\tag{1.51}$$

**(a)** Show that

$$\langle q_f,t_f|q_i,t_i\rangle = F(t_f,t_i)\exp\left[iS_c(q_f,t_f;\ q_i,t_i)\right]\tag{1.52}$$

where $S_c(q_f,t_f;\ q_i,t_i)$ is the action for the classical trajectory, and $F(t_f,t_i)$, being independent of $q_i$ and $q_f$, can be written as

$$F(t_f,t_i)=N\int_{(0,t_i)}^{(0,t_f)}[d\eta(t)]\exp\left\{i\int_{t_i}^{t_f}dt\left[a\dot{\eta}^2+c\eta\dot{\eta}+e\eta^2\right]\right\}.\tag{1.53}$$

Namely, we have the boundary condition $\eta(t)=\eta(t')=0$. Thus $\eta(t)$ can be thought of as the difference between a given $q(t)$ and its classical trajectory.

**(b)** Show that the prefactor $F(t_f, t_i)$ can be expressed in the compact form of

$$F(t_f, t_i) = N'/[\det \hat{A}]^{1/2}, \tag{1.54}$$

with the differential operator $\hat{A} = -a(d^2/dt^2) + c(d/dt) + e$ and $N'$ being a constant.

**Suggestion.** Expand $\eta(t)$ as a series in terms of some orthonormal basis functions $\chi_n(t)$ (with $n = 1, 2, 3, \ldots$):

$$\eta(t) = \sum_n c_n \chi_n(t) \tag{1.55}$$

where $\int_{t_i}^{t_f} \chi_n(t)\chi_m(t) = \delta_{nm}$ and $\chi_n(t_i) = \chi_n(t_f) = 0$. The integration measure $N[d\eta(t)] = N\prod_n d\eta(t_n)$ can be taken as $N\prod_n dc_n$. Thus we can obtain an alternative definition of the path integral as

$$\langle q_f, t_f | q_i, t_i\rangle = N \int \prod_n dc_n \exp iS[q]. \tag{1.56}$$

**Solution to Problem 1.6**

**(a)** The path integral representation of the transition amplitude has the form

$$\langle q_f, t_f | q_i, t_i\rangle = N \int_{(q_i,t_i)}^{(q_f,t_f)} [dq] \exp i \int dt\, [L(q, \dot{q}, t)]\,. \tag{1.57}$$

Let $q_c(t)$ be the solution to the equation of motion

$$\frac{\delta S}{\delta q} = 0 \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.58}$$

with the boundary condition $q_c(t_i) = q_i$ and $q_c(t_f) = q_f$. An arbitrary path $q(t)$ can always be written as $q(t) = q_c(t) + \eta(t)$. Namely, $\eta(t)$ is defined to be the deviation of $q(t)$ from the classical trajectory with the boundary condition $\eta(t_i) = \eta(t_f) = 0$. In terms of the unique classical trajectory and $\eta(t)$, we can express the transition amplitude as

$$\langle q_f, t_f | q_i, t_i\rangle = N \int_{(q_i,t_i)}^{(q_f,t_f)} [d\eta(t)] \exp\{iS[q_c + \eta]\}. \tag{1.59}$$

The action $S$ can be expanded in powers of $\eta(t)$: $S[q_c+\eta] = S[q_c]+S_1+S_2$, where $S_1$ is linear in $\eta(t)$ and $S_2$ is quadratic. Since the classical trajectory, according to the *variational principle*, corresponds to the path with respect to which the action is stationary, we have $S_1 = 0$. Thus

$$S[q_c + \eta] = S[q_c] + \int_{t_i}^{t_f} dt\left[a(t)\dot{\eta}^2 + c(t)\eta\dot{\eta} + e(t)\eta^2\right]. \tag{1.60}$$

$S[q_c]$ is independent of $\eta(t)$. Evidently $S_2$ is independent of $q_c$, hence also of $q_i$ and $q_f$. (This is only true for a quadratic action.) One then has

$$\begin{aligned}\langle q_f, t_f | q_i, t_i\rangle &= \exp(iS[q_c])N\int_{(0,t_i)}^{(0,t_f)}[d\eta(t)] \\ &\quad\times \exp\left\{i\int_{t_i}^{t_f} dt\left[a\dot{\eta}^2 + c\eta\dot{\eta} + e\eta^2\right]\right\} \\ &= F(t_f, t_i)\exp\left[iS_c(q_f, t_f;\ q_i, t_i)\right]\end{aligned} \tag{1.61}$$

which is just the claimed result.

**Remark.** In many physical applications of the path integral formalism it is not necessary to know the prefactor $F(t_f, t_i)$, which does not depend on the coordinates $(q_f, q_i)$.

**(b)** Start from the expression

$$\begin{aligned}F(t_f, t_i) &= N\int[d\eta(t)]\exp\left\{i\int_{t_i}^{t_f} dt\left[a\dot{\eta}^2 + c\eta\dot{\eta} + e\eta^2\right]\right\} \\ &= N\int[d\eta(t)]\exp\left\{i\int_{t_i}^{t_f} dt\,\eta(t)\left[-a\frac{d^2}{dt^2} + c\frac{d}{dt} + e\right]\eta(t)\right\}\end{aligned} \tag{1.62}$$

where to reach the second line we have performed an integration by parts. Now expand $\eta(t)$ in terms of a complete set of orthonormal functions: $\eta(t) = \sum_n c_n\chi_n(t)$ with the condition of $\chi_n(t_i) = \chi_n(t_f) = 0$. We then have

$$F(t_f, t_i) = N\int\prod_n dc_n \exp\left\{i\int_{t_i}^{t_f} dt\,\eta(t)\hat{A}\eta(t)\right\} \tag{1.63}$$

where $\hat{A}$ is the differential operator given in the problem. For convenience, we can choose the orthornormal functions to be the eigenfunctions of $\hat{A}$:

$$\hat{A}\chi_n(t) = \left[-a\frac{d^2}{dt^2} + c\frac{d}{dt} + e\right]\chi_n(t) = k_n\chi_n(t). \tag{1.64}$$

Then

$$\int_{t_i}^{t_f} dt\,\eta(t)\hat{A}\eta(t) = \sum_{n,m}\int dt\, c_n c_m k_m\chi_n(t)\chi_m(t) = \sum_n c_n^2 k_n, \tag{1.65}$$

and the prefactor for the path integral becomes

$$F(t_f, t_i) = N\prod_n\int dc_n\exp\left(i\sum_l c_l^2 k_l\right). \tag{1.66}$$

For each term with $n = l$ we can use the Gaussian integral $\int dx \exp(-ax^2) = (\pi/a)^{1/2}$ to obtain the promised result:

$$F(t_f, t_i) = N \prod_n \left(\frac{i\pi}{k_n}\right)^{1/2} = N' \det A^{-1/2} \tag{1.67}$$

We can check this result by working out explicitly the simple case of a free particle $S = \int_{t_i}^{t_f} dt (m/2)\dot{q}^2$. Thus $A = -(m/2)(d^2/(dt^2)$ and the eigenvalue equation has the form for a simple harmonic oscillator equation, $-(m/2)(d^2\chi_n/dt^2) = k_n\chi_n$, which has the solution of $\chi_n = \alpha_n \sin\omega_n(t - t_i)$, with $\omega_n = (2k_n/m)^{1/2}$. The boundary condition $\omega_n(t_f - t_i) = n\pi$, with $n$ being an integer, implies that the eigenvalues $k_n = (m/2)(n\pi/t_f - t_i)^2$. Thus, the determinant has the value of $\det A^{-1/2} = \prod_n k_n^{-1/2}$. Include the multiplicative factor from the Gaussian integral of $(m/2)(i\pi)^{1/2}$ and choose the normalization factor $N' = [2n/im(t_f - t_i)]^{n/2}$ so that we obtain the expected value (as determined in Problem 1.1) for the prefactor:

$$F(t_f, t_i) = N' \prod_n \left(\frac{i\pi m}{2}\right)^{1/2} \left(\frac{t_f - t_i}{n\pi}\right) = \left[\frac{m}{2\pi i(t_f - t_i)}\right]^{1/2}. \tag{1.68}$$

## 1.7 Spreading of a wave packet

The time-dependent Schrödinger wave function is defined by $\psi(q, t) = \langle q, t|\psi\rangle$.

**(a)** Show that

$$\psi(q_f, t_f) = \int \langle q_f, t_f|q_i, t_i\rangle \psi(q_i, t_i)\, dq_i. \tag{1.69}$$

**(b)** For the free particle, suppose $\psi(q, t = 0)$ is a Gaussian wave packet:

$$\psi(q, 0) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(q-a)^2}{4\sigma^2}\right]. \tag{1.70}$$

Show that it will spread as time evolves:

$$|\psi(q, t)|^2 = \left[\frac{1}{2\pi\sigma^2(t)}\right]^{1/2} \exp\left[-\frac{(q-a)^2}{2\sigma^2(t)}\right] \tag{1.71}$$

where

$$\sigma^2(t) = \sigma^2\left(1 + \frac{t^2}{4m^2\sigma^4}\right). \tag{1.72}$$

**Remark.** One may recall the physical interpretation for this spreading Gaussian wave packet. The initial Gaussian wave function can be thought of as a superposition of plane waves $e^{i\mathbf{p}\cdot\mathbf{x}}$. As they evolve with time, such plane waves acquire a momentum and time-dependent phase $e^{-ip^2t/2m}$, which will make the superposition go out of phase for $t \neq 0$.

**Solution to Problem 1.7**

**(a)** This connection between the initial and final wave functions by the transition amplitude can be obtained simply by inserting a complete set of states in the expression for the wave function,

$$\psi(q_f, t_f) = \langle q_f, t_f|\psi\rangle = \int dq_i \langle q_f, t_f|q_i, t_i\rangle\langle q_i, t_i|\psi\rangle. \tag{1.73}$$

**(b)** Substituting into the above equation the expression for transition amplitude for the free particle case as derived in Problem 1.1 and the wavefunction $\psi(q, 0)$,

$$\langle q, t|q, 0\rangle = \left(\frac{m}{2\pi i t}\right)^{1/2} \exp\left[\frac{im}{2}\frac{(q-q')^2}{t}\right] \tag{1.74}$$

we end up with a Gaussian integral of the form shown in Problem 1.5

$$\begin{aligned}\psi(q, t) &= \int dq' \left(\frac{m}{2\pi i t}\right)^{1/2} \exp\left[\frac{im}{2}\frac{(q-q')^2}{t}\right]\left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(q'-a)^2}{4\sigma^2}\right] \\ &= \left(\frac{m}{2\pi i t}\right)^{1/2}\left(\frac{1}{2\pi\sigma^2}\right)^{1/4}\left[\frac{4\pi i\sigma^2 t}{it + 2m\sigma^2}\right]^{1/2} \exp\left[-\frac{(q-a)^2}{4\sigma^2 + i2t/m}\right].\end{aligned} \tag{1.75}$$

The $\psi^*\psi$ has a simpler expression; it is straightforward to show it checks with the result given in the problem.

## 1.8 Path integral for a harmonic oscillator

The Lagrangian is given by

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2. \tag{1.76}$$

**(a)** Show that the transition amplitude has the form:

$$\begin{aligned}\langle q_f, t_f|q_i, t_i\rangle = &\left[\frac{m\omega}{2\pi i \sin\omega(t_f - t_i)}\right]^{1/2} \exp\left\{\frac{im\omega}{2\sin\omega(t_f - t_i)}\right. \\ &\left.\times\left[(q_f^2 + q_i^2)\cos\omega(t_f - t_i) - 2q_f q_i\right]\right\}.\end{aligned} \tag{1.77}$$

**(b)** Show that for an initial wave packet of the Gaussian form

$$\psi_a(q, 0) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left[-\frac{m\omega}{2}(q-a)^2\right], \tag{1.78}$$

we have

$$|\psi_a(q, t)|^2 = \left(\frac{m\omega}{\pi}\right)^{1/2} \exp\left[-m\omega(q - a\cos\omega t)^2\right]. \tag{1.79}$$

Namely, there is *no* spreading of the wave packet.

**(c)** In general, the transition amplitude, as a Green's function, can be expressed in terms of the energy eigenfunctions as

$$\langle q', t'|q, t\rangle = \sum_n \phi_n(q')\phi_n^*(q)\, e^{iE_n(t'-t)} \tag{1.80}$$

where $\phi_n(q) = \langle q|n\rangle$ and $H|n\rangle = E_n|n\rangle$. Show this and then work out the ground state energy and wave function by taking the limit of $t = 0$ and $t' \to -i\infty$ in the transition amplitude $\langle q', t'|q, t\rangle$.

## Solution to Problem 1.8

**(a)** The action being a quadratic function, the transition amplitude, according to Problem 1.6, has the form of

$$\langle q_f, t_f|q_i, t_i\rangle = F(t_f, t_i)\exp\left[iS_c(q_f, t_f;\ q_i, t_i)\right]. \tag{1.81}$$

Thus we need first to calculate the classical action $S_c$, then the prefactor $F$. Given the Lagrangian we can immediately write down the equation of motion: $\ddot{q} + \omega^2 q = 0$. Its solution corresponds to the classical trajectory: $q_c(t) = A\sin\omega t + B\cos\omega t$ with its coefficients $A$ and $B$ to be fixed by the boundary conditions of $q_c\ (t = t_i \equiv 0) = q_i$ and $q_c\ (t = t_f) = q_f$. We find $B = q_i$ and $A = (q_f - q_i\cos\omega t_f)/\sin\omega t_f$. Thus,

$$q_c(t) = \frac{1}{\sin\omega t_f}\left[q_i\sin\omega(t_f - t) + q_f\sin\omega t\right] \tag{1.82}$$

with the velocity

$$\dot{q}_c(t) = \frac{\omega}{\sin\omega t_f}\left[-q_i\cos\omega(t_f - t) + q_f\cos\omega t\right]. \tag{1.83}$$

The classical action is

$$\begin{aligned} S_c(q_f, t_f;\ q_i, 0) &= \frac{m}{2}\int_0^{t_f} dt\left[\dot{q}_c^2(t) - \omega^2 q_c^2(t)\right] \\ &= \frac{\omega^2}{\sin^2\omega t_f}\frac{m}{2}\int_0^{t_f} dt\,[q_i^2\cos 2\omega(t_f - t). \\ &\quad + q_f^2\cos 2\omega t - 2q_iq_f\cos\omega(t_f - 2t)]. \end{aligned} \tag{1.84}$$

It is straightforward to do the $t$-integrals

$$\begin{aligned} S_c(q_f, t_f;\ q_i, 0) &= \frac{m\omega^2}{2\sin^2\omega t_f}\left[q_i^2\frac{\sin 2\omega t_f}{2\omega} + q_f^2\frac{\sin 2\omega t_f}{2\omega} - 2q_iq_f\frac{\sin\omega t_f}{\omega}\right] \\ &= \frac{m\omega}{2\sin\omega t_f}\left[(q_i^2 + q_f^2)\cos\omega t_f - 2q_iq_f\right]. \end{aligned} \tag{1.85}$$

This is the expression for the classical action that appears in the transition amplitude:

$$\langle q_f, t_f | q_i, t_i \rangle = F(t_f, t_i) \exp\left[ i S_c(q_f, t_f;\ q_i, t_i) \right]. \qquad (1.86)$$

Now we are ready to calculate the prefactor $F(t_f, t_i)$. It can be determined from the condition of

$$\langle q_f, t_f | q_i, t_i \rangle = \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle \qquad (1.87)$$

where we have inserted a complete set of state $\{|q, t\rangle\}$ for a fixed time $t$. Explicitly writing this out, we have

$$\begin{aligned} &F(t_f, t_i) \exp\left[ i S_c(q_f, t_f; q_i, t_i) \right] \\ &\quad = F(t_f, t) F(t, t_i) \int dq \exp[i S_c(q_f, t_f; q, t) + \ i S_c(q, t; q_i, t_i)]. \end{aligned} \qquad (1.88)$$

The integral on the right-hand side has the form

$$\left[ \int_{-\infty}^{+\infty} dq \exp(-Aq^2 + Bq) \right] e^C = \left( \frac{\pi}{A} \right)^{1/2} \exp\left( \frac{B^2}{4A} + C \right) \qquad (1.89)$$

where

$$A = \frac{m\omega}{2i} \left[ \frac{\cos\omega(t_f - t)}{\sin\omega(t_f - t)} + \frac{\cos\omega(t - t_i)}{\sin\omega(t - t_i)} \right]. \qquad (1.90)$$

For our purpose of calculating the prefactor, there is no need to work out the $Bq+C$ term as it only contributes to the exponent, which must match the $i S_c(q_f, t_f; q_i, t_i)$ factor on the left-hand side of the equation. Thus with the presumed cancellation (*check this*) of the resultant exponential factors on both sides of the equation, the only relevant Gaussian integral is

$$\int dq \exp Aq^2 = \left( \frac{\pi}{A} \right)^{1/2} = \left[ \frac{2\pi i \sin\omega(t_f - t) \sin\omega(t - t_i)}{m\omega \sin\omega(t_f - t_i)} \right]^{1/2}. \qquad (1.91)$$

This means that the prefactors have the following relation:

$$\begin{aligned} &\left[ \frac{m\omega}{2\pi i} \right]^{1/2} \left[ \sin\omega(t_f - t_i) \right]^{1/2} F(t_f, t_i) \\ &\quad = [\sin\omega(t_f - t)]^{1/2} F(t_f, t) \cdot [\sin\omega(t - t_i)]^{1/2} F(t, t_i). \end{aligned} \qquad (1.92)$$

From the above equation we can deduce the desired result:

$$F(t_f, t_i) = \left[ \frac{m\omega}{2\pi i \sin\omega(t_f - t_i)} \right]^{1/2}. \qquad (1.93)$$

**(b)** From Problems 6 and 7, we have the relation

$$\psi(q,t) = \int dq' \langle q,t|q',0\rangle \psi(q',0) \tag{1.94}$$

where

$$\langle q,t|q',0\rangle = \left[\frac{m\omega}{2\pi i \sin\omega t}\right]^{1/2} \exp\left\{\frac{im\omega}{2\sin\omega t}\left[(q^2+q'^2)\cos\omega t - 2qq'\right]\right\}. \tag{1.95}$$

and, as is given,

$$\psi_a(q',0) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left[-\frac{m\omega}{2}(q'-a)^2\right]. \tag{1.96}$$

Putting them together,

$$\begin{aligned}\psi_a(q,t) = \left(\frac{m\omega}{\pi}\right)^{1/4}\left(\frac{m\omega}{2\pi i\sin\omega t}\right)^{1/2}\int dq' \\ \times \exp\left\{\frac{im\omega}{2\sin\omega t}\left[(q^2+q'^2)\cos\omega t - 2qq'\right] - \frac{m\omega}{2}(q'-a)^2\right\}.\end{aligned} \tag{1.97}$$

The exponential integrand having a quadratic function of $q'$: $\{\cdots\} = -Aq'^2 + Bq' + C$ as its exponent

$$\begin{aligned} A &= \frac{-im\omega\cos\omega t}{2\sin\omega t} + \frac{m\omega}{2} = \frac{-im\omega}{2\sin\omega t}\, e^{i\omega t} \\ B &= \frac{-im\omega}{\sin\omega t}(q + ia\sin\omega t) \\ C &= \frac{im\omega\cos\omega t}{2\sin\omega t}q^2 - \frac{m\omega}{2}a^2, \end{aligned} \tag{1.98}$$

the integral is of the Gaussian type discussed in part (a) and yields the result of $(\pi/A)^{1/2}\exp((B^2/4A)+C)$. We obtain the final wave function

$$\psi_a(q,t) = \left(\frac{m\omega}{\pi}\right)^{1/4}\left[\frac{m\omega}{2\pi i\sin\omega t}\,\frac{2\pi\sin\omega t}{-im\omega}\,e^{-i\omega t}\right]^{1/2}\exp[\cdots] \tag{1.99}$$

where the exponent has real and imaginary parts:

$$\begin{aligned}[\cdots] &= \frac{-im\omega\, e^{-i\omega t}}{2\sin\omega t}(q+ia\sin\omega t)^2 + \frac{im\omega\cos\omega t}{2\sin\omega t}q^2 - \frac{m\omega}{2}a^2 \\ &= -\frac{m\omega}{2}(q-a\cos\omega t)^2 + i\frac{m\omega}{2}\sin\omega t\,(2aq + a^2\cos\omega t).\end{aligned} \tag{1.100}$$

In this way we obtain the wave function

$$\begin{aligned}\psi_a(q,t) = \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-i\omega t/2}\exp\left[-\frac{m\omega}{2}(q-a\cos\omega t)^2\right] \\ \times \exp i\left[\frac{m\omega}{2}\sin\omega t\,(2aq + a^2\cos\omega t)\right],\end{aligned} \tag{1.101}$$

and an expression for $|\psi_a(q,t)|^2$ just as that quoted in the problem. There is no spread of the wave packet because the original Gaussian wave function is an eigenfunction of the SHO Hamiltonian, and the time evolution comes in only as an overall phase.

**(c)** We will first express the transition amplitude in terms of the energy eigenfunctions:

$$\begin{aligned}\langle q', t'|q, t\rangle &= \langle q'|\exp[iH(t'-t)]|q\rangle = \sum_n \langle q'|\exp[iH(t'-t)]|n\rangle\langle n|q\rangle \\ &= \sum_n e^{iE_n(t'-t)}\langle q'|n\rangle\langle n|q\rangle = \sum_n \phi_n(q')\phi_n^*(q)\, e^{iE_n(t'-t)}.\end{aligned}$$

Setting $t = 0$ for convenience, it is clear that in the limit of $t' \equiv -iT$ with $T \to \infty$ this sum is dominated by the ground state $|0\rangle$ contribution:

$$\lim_{T\to\infty}\langle q', -iT|q, 0\rangle = \lim_{T\to\infty}\phi_0(q')\phi_0^*(q)\, e^{-E_0 T}. \tag{1.102}$$

This should be compared to the limiting expression for the transition amplitude obtained in part (a)

$$\langle q', t'|q, 0\rangle = \left(\frac{m\omega}{2\pi i \sin\omega t'}\right)^{1/2}\exp\left\{\frac{im\omega}{2\sin\omega t'}\left[(q'^2+q^2)\cos\omega t' - 2q'q\right]\right\}, \tag{1.103}$$

Noting that both $\cos\omega t'$ and $i\sin\omega t'$ increase as $\frac{1}{2}e^{\omega T}$ in this limit:

$$\begin{aligned}\langle q', -iT|q, 0\rangle &= \left(\frac{m\omega}{\pi e^{\omega T}}\right)^{1/2}\exp\left\{\frac{-m\omega}{e^{\omega T}}\left[(q'^2+q^2)\tfrac{1}{2}e^{\omega T} - 2q'q\right]\right\} \\ &= \left(\frac{m\omega}{\pi}\right)^{1/2}\exp\left\{\frac{-m\omega}{2}[(q'^2+q^2)]\right\}e^{-\omega T/2}.\end{aligned} \tag{1.104}$$

Compare the expressions in eqns (1.102) and (1.104) and we obtain the ground state energy and wave function as

$$E_0 = \frac{1}{2}\omega, \qquad \phi_0(q) = \left(\frac{m\omega}{\pi}\right)^{1/4}\exp\left(\frac{-m\omega}{2}q^2\right). \tag{1.105}$$

## 1.9 Path integral for a partition function

Show that the partition function of canonical ensemble $Z = Tr(e^{-\beta H})$ with $\beta = (kT)^{-1}$ and $H$ the Hamiltonian, for a simple case of one degree of freedom system, can be written as a path integral representation as

$$Z = \int dq_o \int [dq]\exp\left(i\int_0^{-i\beta} dt\,\Lambda(q,\dot{q})\right) \tag{1.106}$$

where $q_o = q(t = 0) = q(-i\beta)$ and $\Lambda(q, \dot{q})$ is the Lagrangian in Euclidean time $\tau = it$.

**Solution to Problem 1.9**

The trace in the partition function can be written as the sum of the matrix elements of the $e^{-\beta H}$ operator between the eigenstates of the $q$ operator,

$$Z = \sum_{q_o} \langle q_o | e^{-\beta H} | q_o \rangle. \tag{1.107}$$

Compare this with the path integral representation of the transition amplitude, cf. in particular CL-eqn (1.47),

$$\begin{aligned}\langle q_f, t_f | q_i, t_i \rangle &= \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle \\ &= \int [dq] \int [dp] \exp\left( i \int_{t_i}^{t_f} dt [p\dot{q} - H(q,p)] \right).\end{aligned} \tag{1.108}$$

We see that the partition function in eqn (1.107) can be viewed as the special case of the path integral representation in eqn (1.108) with the restriction

$$t_f - t_i = -i\beta \qquad \text{or} \qquad t_i = 0 \quad \text{and} \quad t_f = -i\beta \tag{1.109}$$

and the initial and final position identified $q_f = q_i = q_o$. Namely, the path $q(t)$ is periodical

$$q(t_f) = q(t_i) = q_o. \tag{1.110}$$

For convenience, we can introduce the new variable $\tau = it$. The integral in eqn (1.108) becomes

$$i \int_0^{-i\beta} dt \, [p\dot{q} - H(q,p)] = \int_0^{\beta} d\tau \left[ ip\frac{dq}{d\tau} - H(q,p) \right] \tag{1.111}$$

and therefore

$$Z = \int [dq] \int [dp] \exp\left\{ \int_0^{\beta} d\tau \left[ ip\frac{dq}{d\tau} - H(q,p) \right] \right\}. \tag{1.112}$$

For most cases, $H = (p^2/2m) + V(q)$, the momentum integral is Gaussian

$$\begin{aligned}&\int dp \exp\left\{ \int_0^{\beta} d\tau \left[ -\frac{1}{2m}p^2 + i\frac{dq}{d\tau}p - V \right] \right\} \\ &\quad = N' \exp\left\{ \int_0^{\beta} d\tau \left[ -\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 - V \right] \right\}.\end{aligned} \tag{1.113}$$

And we have the partition function

$$Z = N \int [dq] \exp\left\{ -\int_0^{\beta} d\tau \left[ \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V \right] \right\} \tag{1.114}$$

where the combination $-[(m/2)(dq/d\tau)^2 + V]$ is just the usual Euclidean Lagrangian $\Lambda(q, \dot{q})$. The constant $N$ is independent of temperature (hence has no physical significance). The integration is over all the periodic paths with the boundary path $q_o = q(0) = q(\beta)$.

From this problem we see that the partition function can be obtained from the usual path integral method through the steps (i) set $t_i = 0$, and $t_f = -i\beta$, and $q(t_f) = q(t_i) = q_o$; and (ii) integrate over $q_o$.

## 1.10 Partition function for an SHO system

The partition function for the case of the simple harmonic oscillator $V(q) = (m\omega/2)q^2$ can be obtained as follows

$$Z = tr(e^{-\beta H}) = \sum_n e^{-\beta(n+(1/2))\omega} = \left(2\sinh\frac{\omega\beta}{2}\right)^{-1}. \tag{1.115}$$

Now use the path integral method, as outlined in Problem 1.9, to recover this result in two ways:

**(a)** by making use of the SHO result of Problem 1.8, then performing the integration over boundary values of $q_o = q(t_f) = q(t_i)$ as a simple Gaussian integral, or

**(b)** by using the approach of calculating the path integral as indicated in Problem 1.6: $Z \propto \det A^{-1/2}$ with $A$ being the appropriate operator for the SHO case.

**Solution to Problem 1.10**

**(a)** From Problem 1.8, we have obtained the SHO transition amplitude

$$\begin{aligned}\langle q_f, t_f | q_i, t_i\rangle = \left[\frac{m\omega}{2\pi i \sin\omega(t_f - t_i)}\right]^{1/2} \exp\Bigg\{ &\frac{im\omega}{2\sin\omega(t_f - t_i)} \\ &\times \left[(q_i^2 + q_f^2)\cos\omega(t_f - t_i) - 2q_f q_i\right]\Bigg\}.\end{aligned} \tag{1.116}$$

To get the partition function by following the method given in Problem 1.9, we set $t_i = 0$ and $q_f = q_i = q_o$; the exponent in the above equation becomes

$$\begin{aligned}\{\cdots\} &= \frac{im\omega}{2\sin\omega t_f} 2q_o^2(\cos\omega t_f - 1) = \frac{-2im\omega\sin^2(\omega t_f/2)}{\sin\omega t_f} q_o^2 \\ &= \frac{-im\omega\sin(\omega t_f 2)}{\cos(\omega t_f/2)} q_o^2 = -im\omega q_o^2 \tan\frac{\omega t_f}{2}.\end{aligned} \tag{1.117}$$

By integrating over $q_o$ and setting $t_f = -i\beta$, we then obtain

$$\begin{aligned}Z &= \left(\frac{m\omega}{2\pi i\sin\omega t_f}\right)^{1/2} \int dq_o \exp\left[-\left(im\omega\tan\frac{\omega t_f}{2}\right) q_o^2\right] \\ &= \left(\frac{m\omega}{2\pi i\sin\omega t_f}\right)^{1/2} \left(\frac{\pi}{im\omega\tan(\omega t_f/2)}\right)^{1/2} = \left(2i\sin\frac{\omega t_f}{2}\right)^{-1} \\ &= \left(2\sinh\frac{\omega\beta}{2}\right)^{-1}.\end{aligned} \tag{1.118}$$

**(b)** We start with the path integral representation of the partition function as given in eqn (1.114):

$$\begin{aligned} Z &= N \int [dq] \exp\left\{-\int_0^\beta d\tau \left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + \frac{\omega^2}{2}q^2\right]\right\} \\ &= N' \int [dq] \exp\left[-\frac{1}{2}\int_0^\beta d\tau q \left(-\frac{d^2}{d\tau^2} + \omega^2\right) q\right] \\ &= N''[\det A]^{-1/2} \end{aligned} \tag{1.119}$$

where $A = -(d^2/d\tau^2) + \omega^2$ and $\det A = \prod_n a_n$ with $a_n$ being the eigenvalues of $A$ on the space of periodic functions $f(\tau) = f(\tau + \beta)$. We expand this periodic function as $f(\tau) = \sum_n c_n e^{in2\pi\tau\beta^{-1}}$, then the eigenvalues are $a_n = \omega^2 + 4\pi^2 n^2 \beta^{-2}$ for the eigenfunction $e^{in2\pi\tau\beta^{-1}}$. In order to evaluate such a determinant of a series, we first take the logarithm of the determinant

$$\ln \det A = \sum_n \ln(\omega^2 + 4\pi^2 n^2 \beta^{-2}). \tag{1.120}$$

To evaluate this series, we note that

$$\ln \det A = \int_0^\omega d\chi \frac{d \ln \det A}{d\chi} = 2\int_0^\omega \chi\, d\chi \frac{d \ln \det A}{d\chi^2}. \tag{1.121}$$

In the integrand we have substituted $\omega$ by $\chi$.

$$\begin{aligned} \frac{d}{d\chi^2} \ln \det A &= \sum_{n=-\infty}^{+\infty} (\chi^2 + 4\pi^2 n^2 \beta^{-2})^{-1} \qquad (1.122) \\ &= \chi^{-2} + 2\sum_{n=1}^{+\infty}(\chi^2 + 4\pi^2 n^2\beta^{-2})^{-1} \\ &= \chi^{-2} + \frac{2\beta^2}{4\pi^2}\sum_{n=1}^{+\infty}\left(n^2 + \frac{\beta^2\chi^2}{4\pi^2}\right)^{-1} \\ &= \chi^{-2}\pi\frac{\beta\chi}{2\pi}\coth\left(\pi\frac{\beta\chi}{2\pi}\right) = \frac{\beta}{2\chi}\coth\frac{\beta\chi}{2} \end{aligned}$$

where we have used the identity

$$\pi y \coth \pi y = 1 + 2y^2 \sum_{n=1}^{+\infty}(n^2 + y^2)^{-1}. \tag{1.123}$$

In this way

$$\ln \det A = \int_0^\omega \beta\, d\chi \coth\frac{\beta\chi}{2} = 2\ln\left(\sinh\frac{\beta\chi}{2}\right) \tag{1.124}$$

or $\det A = (\sinh(\beta\chi/2))^2$. Thus

$$Z = N\left(\sinh\frac{\beta\chi}{2}\right)^{-1} \tag{1.125}$$

which is in agreement with the result obtained in (a) when we recall that the temperature independent constant in front has no physical significance.

## 1.11 Non-standard path-integral representation

Consider the Lagrangian with a position-dependent 'mass'

$$L = \frac{1}{2} f(q)\dot{q}^2. \tag{1.126}$$

Thus the canonical momentum is $p = f(q)\dot{q}$ and the Hamiltonian $H = (p^2/2f(q))$. Show that the path-integral representation of the transition amplitude has the form

$$\langle q', t'|q, t\rangle = N' \int [dq] \exp\left\{i \int dt \left[L(q, \dot{q}) - \frac{i}{2}\delta(0) \ln f(q)\right]\right\}. \tag{1.127}$$

**Suggestion.** One can start with the expression of the transition amplitude

$$\begin{aligned}\langle q', t'|q, t\rangle &= N \int [dq] \int [dp] \exp\left\{i \int_t^{t'} dt\, [p\dot{q} - H(q, p)]\right\}\\ &= \lim_{n\to\infty} \int \frac{dp_1}{2\pi} \cdots \frac{dp_n}{2\pi} dq_1 \cdots dq_n \exp\left\{i \sum_i^n \delta t \left[p_i \left(\frac{q_i - q_{i-1}}{\delta t}\right)\right.\right.\\ &\quad \left.\left. - H\left(p_i, \frac{q_i + q_{i-1}}{2}\right)\right]\right\}\end{aligned} \tag{1.128}$$

and then explicitly perform the momentum integrations.

**Remark.** This is the counter-example, first given by Lee and Yang (1962), showing that path-integral representation is not always of the form

$$\langle q', t'|q, t\rangle = N' \int [dq] \exp\left\{i \int dt\, L(q, \dot{q})\right\}. \tag{1.129}$$

**Solution to Problem 1.11**

For the given Lagrangian, we have

$$\begin{aligned}\langle q', t'|q, t\rangle &= N \int [dq] \int [dp] \exp\left\{i \int_t^{t'} dt \left[p\dot{q} - \frac{p^2}{2f(q)}\right]\right\}\\ &= \lim_{n\to\infty} \int \frac{dp_1}{2\pi} \cdots \frac{dp_n}{2\pi} dq_1 \cdots dq_{n-1}\\ &\quad \times \exp\left\{i \sum_i^n \delta t \left[p_i \left(\frac{q_i - q_{i-1}}{\delta t}\right) - \frac{p_i^2}{2f(q_i)}\right]\right\}.\end{aligned} \tag{1.130}$$

The momentum integrals are of the Gaussian type $\int_{-\infty}^{+\infty} dx \exp(-Ax^2 + B) = (\pi/A)^{1/2} \exp(B^2/4A)$:

$$\begin{aligned}I_i &= \int_0^{\infty} dp_i \exp\left\{i\delta t \left[p_i \left(\frac{q_i - q_{i-1}}{\delta t}\right) - \frac{p_i^2}{2f(q_i)}\right]\right\}\\ &= \left[\frac{2\pi f(q_i)}{i\delta t}\right]^{1/2} \exp\left[-\frac{2f(q_i)}{4i\delta t}(q_i - q_{i-1})^2\right].\end{aligned} \tag{1.131}$$

Since we are eventually interested in expressing the $[dq]$ integrand in the form of an exponential, we will now write the position-dependent part of the prefactor in the exponential form $[f(q)]^{1/2} = \exp\left\{\frac{1}{2}\ln f(q)\right\}$. In this way the above integral becomes

$$I_i = \left(\frac{2\pi}{i\delta t}\right)^{1/2} \exp\left\{i\left[\frac{f(q_i)}{2}\left(\frac{q_i - q_{i-1}}{\delta t}\right)^2 - \frac{i\ln f(q_i)}{2\delta t}\right]\delta t\right\}. \tag{1.132}$$

This means that the path-integral representation can be written

$$\begin{aligned}\langle q', t'|q, t\rangle &= \lim_{n\to\infty}(2\pi i\delta t)^{-n/2}\int dq_1\cdots dq_{n-1}\\ &\quad\times\exp\left\{i\sum_{i=1}^{n}\left[\frac{f(q_i)}{2}\left(\frac{q_i - q_{i-1}}{\delta t}\right)^2 - \frac{i\ln f(q_i)}{2\delta t}\right]\delta t\right\}\\ &= N\int[dq]\exp\left\{i\int\left[\frac{f(q)}{2}\dot{q}^2 - \frac{i\ln f(q)}{2}\delta(0)\right]dt\right\}.\end{aligned} \tag{1.133}$$

This is the claimed result. To get to the last line we have used the expression for Dirac's delta function as

$$\delta(0) = \lim_{\delta t\to 0}\frac{1}{\delta t}. \tag{1.134}$$

## 1.12 Weyl ordering of operators

**Notes on operator ordering**

For the simple system of which the Hamiltonian in the form

$$H_c(p, q) = \frac{p^2}{2m} + V(q) \tag{1.135}$$

has no terms that depend on both $p$ and $q$, the quantization is straightforward: just replace the classical variables $(p, q)$ by operators $(\hat{p}, \hat{q})$. Thus the quantum Hamiltonian operator is unique:

$$\hat{H}(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{q}), \tag{1.136}$$

and we have the path-integral representation of the matrix element

$$\langle q_{i+1}|\hat{H}(\hat{p}, \hat{q})|q_i\rangle = \int\frac{dp_i}{2\pi}H_c\left(p_i, \frac{q_{i+1} + q_i}{2}\right)e^{ip_i(q_{i+1}-q_i)}. \tag{1.137}$$

For the more general case of which the Hamiltonian function $H_c(p, q)$ can have mixed terms of $p$ and $q$, for example $p^2q^2$, the quantized theory is not unique. Each of the choices: $\hat{p}^2\hat{q}^2$, $\hat{q}\hat{p}^2\hat{q}$, $\hat{p}\hat{q}^2\hat{p}$, and $\hat{q}^2\hat{p}^2$ will have the same classical limit. If these choices are non-equivalent, experimental measurements will, in principle, pick out the correct choice. Problem 12 will illustrate the point that for a Hamiltonian containing terms of the type $q^n p^m$, the path integral quantization corresponds to, in some sense, the most symmetric ordering, called *Weyl ordering*. It is defined to be the average of all possible orderings of $p$s and $q$s. For instance,

$$\begin{aligned}
(pq^n)_W &\equiv \frac{1}{n+1}\sum_{l=0}^{n}(q^l pq^{n-l}) = \frac{1}{n+1}(q^n p + q^{n-1}pq + \cdots + pq^n) \\
(p^2q^n)_W &\equiv \frac{2}{(n+1)(n+2)}\sum_{l,m=0}^{n} q^m pq^l pq^{n-l-m}.
\end{aligned} \tag{1.138}$$

An instructive discussion of Weyl ordering can be found in the book by Lee (1990).

**Remark.** To do Problem 1.12, you may find the following identities useful:

$$\sum_{l=1}^{n} l = \frac{1}{2}n(n+1) \tag{I-1}$$

$$\sum_{l=1}^{n} l^2 = \frac{1}{6}n(n+1)(2n+1) \tag{I-2}$$

$$\sum_{l=1}^{n} l^3 = \left[\frac{1}{2}n(n+1)\right]^2 \tag{I-3}$$

and

$$2^n = \sum_{l=1}^{n}\binom{n}{l} \tag{I-4}$$

$$n2^{n-1} = \sum_{l=1}^{n} l\binom{n}{l} \tag{I-5}$$

$$n(n-1)2^{n-2} = \sum_{l=1}^{n} l(l-1)\binom{n}{l} \tag{I-6}$$

where $\binom{n}{l} = (n!/l!(n-l)!)$ are the binomial coefficients. Can you prove these identities?

**Suggestion.** One approach to the first three relations will be to use the equalities $\sum_{l=0}^{n}(l-1)^m - \sum_{l=0}^{n} l^m = (n+1)^m$ for $m = 2, 3, 4$. We note that the left-hand sides are just different combinations of $\sum_{l=0}^{n} l^k$ with $k = 1, 2, 3$. The last three identities (I-4, I-5, and I-6) simply follow from the successive differentiations $(d^n/dx^n)(1+x)^n$ at $x = 1$ for $m = 0, 1, 2$.

**Problems on Weyl ordering**

**(a)** Show that the Weyl ordering of operators $p$ and $q$ can be written as

$$(pq^n)_W \equiv \frac{1}{n+1}(q^n p + q^{n-1}pq + \cdots + pq^n)$$
$$= \frac{1}{2^n}\sum_{l=1}^{n} \frac{n!}{l!(n-l)!} q^l p q^{n-l}, \qquad (1.139)$$

with the matrix element of

$$\langle q_{i+1}|(pq^n)_W|q_i\rangle = \left(\frac{q_{i+1}+q_i}{2}\right)^n \langle q_{i+1}|p|q_i\rangle. \qquad (1.140)$$

**(b)** Show the Weyl ordering of the operator product with two powers of $p$:

$$(p^2 q^n)_W \equiv \frac{2}{(n+1)(n+2)} \sum_{l,m=1}^{n} q^m p q^l p q^{n-l-m}$$
$$= \frac{1}{2^n}\sum_{l=1}^{n} \frac{n!}{l!(n-l)!} q^l p^2 q^{n-l}, \qquad (1.141)$$

which leads to the matrix element of

$$\langle q_{i+1}|(p^2 q^n)_W|q_i\rangle = \left(\frac{q_{i+1}+q_i}{2}\right)^n \langle q_{i+1}|p^2|q_i\rangle. \qquad (1.142)$$

**Remark.** According to eqns (1.140) and (1.142), the matrix elements of $q$ in the Weyl ordering are just of the form $(q_{i+1}+q_i)/2$ as prescribed in the path-integral formalism.

**Solution to Problem 1.12**

**(a)** Before working out the situation for general $n$, let us first consider the simplest non-trivial case of $n = 2$:

$$(pq^2)_W \equiv \tfrac{1}{3}(q^2 p + qpq + pq^2) = pq^2 + iq \qquad (1.143)$$

where we have used the commutation relations $[q^m, p] = imq^{m-1}$ to move $q$s to the right of $p$s. The right-hand side is shown to be just the claimed result (as given in the problem) by further application of these commutation relations:

$$\frac{1}{2^2}(q^2 p + 2qpq + pq^2) = \frac{1}{4}\left[(pq^2 + 2iq) + 2(pq^2 + iq) + pq^2\right]$$
$$= pq^2 + iq. \qquad (1.144)$$

The matrix element then has the simple structure:

$$\begin{aligned}\langle q_{i+1}|(pq^2)_W|q_i\rangle &= \frac{1}{2^2}\langle q_{i+1}|q^2p + 2qpq + pq^2|q_i\rangle \\ &= \frac{1}{2^2}(q_{i+1}^2 + 2q_{i+1}q_i + q_i^2)\langle q_{i+1}|p|q_i\rangle \\ &= \left(\frac{q_{i+1}+q_i}{2}\right)^2 \langle q_{i+1}|p|q_i\rangle. \end{aligned} \tag{1.145}$$

Now let us repeat the above steps for the general $n$ situation,

$$\begin{aligned}(pq^n)_W &\equiv \frac{1}{n+1}\sum_{l=0}^{n}(q^l pq^{n-l}) \\ &= \frac{1}{n+1}\sum_{l=0}^{n}(pq^n + ilq^{n-1}) \\ &= pq^n + \frac{i}{2}nq^{n-1}, \end{aligned} \tag{1.146}$$

where to reach the last line we have used the identity (I-1). This result is the same as given in the problem, eqn (1.139), because,

$$\begin{aligned}\frac{1}{2^n}\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}q^l pq^{n-l} &= \frac{1}{2^n}\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}(pq^n + ilq^{n-1}) \\ &= \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}(pq^n + ilq^{n-1}) \\ &= pq^n + \frac{i}{2}nq^{n-1} \end{aligned} \tag{1.147}$$

where we have used (I-4) and (I-5). The combination of eqns (1.146) and (1.147) yields the claimed result:

$$(pq^n)_W = \frac{1}{2^n}\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}q^l pq^{n-l}. \tag{1.148}$$

The general matrix element can now be written as

$$\begin{aligned}\langle q_{i+1}|(pq^n)_W|q_i\rangle &= \frac{1}{2^n}\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\langle q_{i+1}|q^l pq^{n-l}|q_i\rangle \\ &= \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}q_{i+1}^l q_i^{n-l}\langle q_{i+1}|p|q_i\rangle \\ &= \left(\frac{q_{i+1}+q_i}{2}\right)^n \langle q_{i+1}|p|q_i\rangle. \end{aligned} \tag{1.149}$$

**(b)** Before working out the general $n$ situation, let us first consider the simplest non-trivial case of $n = 2$:

$$(p^2q^2)_W \equiv \tfrac{1}{6}(q^2p^2 + qpqp + qp^2q + pq^2p + pqpq + p^2q^2). \qquad (1.150)$$

The right-hand side can then be rearranged by using the commutators

$$\left[q^m, p^2\right] = im(q^{m-1}p + pq^{m-1}), \qquad (1.151)$$

so that

$$\begin{aligned}(p^2q^2)_W &= \tfrac{1}{6}\Big\{\left[p^2q^2 + 2i(qp + pq)\right] + [(pq + i)qp] \\ &\qquad + [(pq + i)pq] + \left[p^2q^2 + 2ipq\right] + \left[p^2q^2 + ipq\right] + p^2q^2\Big\} \\ &= \tfrac{1}{6}(6p^2q^2 + 12ipq - 3) = p^2q^2 + 2ipq - \tfrac{1}{2}. \end{aligned} \qquad (1.152)$$

This last expression can be shown to be just $(q^2p^2 + 2qp^2q + p^2q^2)/4$ which is eqn (1.141) with $n = 2$:

$$\begin{aligned}\tfrac{1}{4}(q^2p^2 + 2qp^2q + p^2q^2) &= \tfrac{1}{4}\left[(p^2q^2 + 2ipq + 2iqp)\right. \\ &\qquad \left. +2(p^2q + 2ip)q + p^2q^2\right] \\ &= \tfrac{1}{4}\left[4p^2q^2 + 8ipq - 2\right] = p^2q^2 + 2ipq - \tfrac{1}{2}. \end{aligned} \qquad (1.153)$$

Thus

$$(p^2q^2)_W = \frac{1}{2^2}(q^2p^2 + 2qp^2q + p^2q^2), \qquad (1.154)$$

and

$$\langle q_{i+1}|(p^2q^2)_W|q_i\rangle = \left(\frac{q_{i+1} + q_i}{2}\right)^2 \langle q_{i+1}|\, p^2|q_i\rangle. \qquad (1.155)$$

Now let us repeat the above steps for the general $n$ situation.

$$\begin{aligned}(p^2q^n)_W &\equiv \frac{2}{(n+1)(n+2)} \sum_{l,m=0}^{n} q^l pq^m pq^{n-l-m} \\ &= \frac{2}{(n+1)(n+2)} \sum_{l,m=0}^{n} (pq^l + ilq^{l-1})q^m pq^{n-l-m} \\ &= \frac{2}{(n+1)(n+2)} \sum_{l,m=0}^{n} (pq^{l+m}pq^{n-l-m} + ilq^{l+m-1}pq^{n-l-m}) \end{aligned}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{l,m=0}^{n} \Big\{ p \left[ pq^{l+m} + i(l+m)q^{l+m-1} \right] q^{n-l-m}.$$

$$+ il \left[ pq^{l+m-1} + i(l+m-1)q^{l+m-2} \right] q^{n-l-m} \Big\}$$

$$= \frac{2}{(n+1)(n+2)}$$

$$\times \sum_{l,m=0}^{n} \left[ p^2 q^n + i(2l+m)pq^{n-1} - l(l+m-1)q^{n-2} \right]. \tag{1.156}$$

We will discuss separately the three terms on the right-hand side of this equation. In the first term we have the sum

$$\sum_{l,m=0}^{n} = \sum_{l=0}^{n} \sum_{m=0}^{n-l} = \sum_{l=0}^{n} (n-l+1) = \sum_{l=0}^{n} (n+1) - \sum_{l=0}^{n} l$$

$$= (n+1)^2 - \tfrac{1}{2} n(n+1) = \tfrac{1}{2}(n+1)(n+2) \tag{1.157}$$

where the identity (I-1) has been used. We then evaluate the sum in the second term of eqn (1.156):

$$\sum_{l,m=0}^{n} (2l+m) = 2 \sum_{m=0}^{n} \sum_{l=0}^{n-m} l + \sum_{m=0}^{n} m \sum_{l=0}^{n-m}$$

$$= 2 \sum_{m=0}^{n} \tfrac{1}{2}(n-m)(n-m+1) + \sum_{m=0}^{n} m(n-m+1)$$

$$= \sum_{m=0}^{n} [n(n+1) - nm] = n(n+1)^2 - \tfrac{1}{2} n^2 (n+1)$$

$$= \tfrac{1}{2} n(n+1)(n+2), \tag{1.158}$$

where (I-1) has been used. To evaluate the sum in the third term of eqn (1.156), we will need to use all three identities (I-1, I-2, and I-3):

$$\sum_{l,m=0}^{n} l(l+m-1)$$

$$= \sum_{l=0}^{n} l \sum_{m=0}^{n-l} m + \sum_{l=0}^{n} l^2 \sum_{m=0}^{n-l} - \sum_{l=0}^{n} l \sum_{m=0}^{n-l}$$

$$= \sum_{l=0}^{n} \left[ l \tfrac{1}{2}(n-l)(n-l+1) + (l^2 - l)(n-l+1) \right]$$

$$= \frac{1}{2} \sum_{l=0}^{n} [-3l^3 + 3(n+1)l^2 - (n+2)l]$$

$$= \frac{1}{2}\left[-3\frac{n^2(n+1)^2}{4} + 3(n+1)\frac{n(n+1)(2n+1)}{6} - (n+2)\frac{n(n+1)}{2}\right]$$
$$= \frac{1}{8}n(n+1)(n+2)(n-1). \tag{1.159}$$

Substituting the results from eqns (1.157) to (1.159) into eqn (1.156), we have

$$(p^2q^n)_W = p^2q^n + inpq^{n-1} - \tfrac{1}{4}n(n-1)q^{n-2}. \tag{1.160}$$

We next show that the right-hand side is equal to $(1/2^n)\sum_{l=0}^{n}\binom{n}{l}q^l p^2 q^{n-l}$:

$$\begin{aligned}
&\frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}q^l p^2 q^{n-l}\\
&= \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}\left[p^2q^l + il(q^{l-1}p + pq^{l-1})\right]q^{n-l}\\
&= \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}\left[p^2q^n + il(q^{l-1}pq^{n-l} + pq^{n-1})\right]\\
&= \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}\left[p^2q^n + 2ilpq^{n-1} - l(l-1)q^{n-2}\right].
\end{aligned} \tag{1.161}$$

Using the identities (I-4, I-5, and I-6), we have

$$\frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}q^l p^2 q^{n-l} = p^2q^n + inpq^{n-1} - \frac{1}{4}n(n-1)q^{n-2}. \tag{1.162}$$

Comparing eqns (1.160) and (1.162) we see the relation of eqn (1.141) is satisfied.

## 1.13 Generating functional for a scalar field

The generating functional for the free scalar field $\phi(x)$ is of the form

$$W_o[J] = \int [d\phi]\exp\left\{i\int d^4x\,\mathcal{L}_J\right\} \tag{1.163}$$

where the Lagrangian density with an external c-number source $J(x)$ is given by

$$\mathcal{L}_J = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 + J\phi. \tag{1.164}$$

**(a)** Show that such an $\mathcal{L}_J$ leads to the equation of motion $(\Box + \mu^2)\phi = J$ with a classical solution that can be obtained by the usual Green's function method:

$$\phi_c(x) = -\int d^4y\,\Delta_F(x-y)J(y). \tag{1.165}$$

The Green's function $\Delta_F(x-y)$ is the Feynman propagator for the scalar field in position space:

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-\mu^2+i\varepsilon}. \tag{1.166}$$

**(b)** Show that, by a change of variable $\phi(x) = \phi_c(x) + \eta(x)$ in the Lagrangian density, the generating functional can be expressed in the form

$$W_o[J] = N \exp\left\{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)\right\} \tag{1.167}$$

where $N$ is a constant, independent of $J$.

**Remark.** One way to understand the $i\varepsilon$ prescription in the expression (1.166) for the propagator $\Delta_F(x-y)$ is to note that the path integral expression in (1.163) is not well defined because of the oscillatory behaviour of $\exp\left\{i\int d^4x\,\mathcal{L}_J\right\}$ for a real $\mathcal{L}_J$. In principle, we have to go over to the Euclidean space–time $t = i\tau$ in order to convert this oscillatory behaviour into a damping one. We then return to the Minkowski space by the method of analytic continuation. However, a much simpler approach that will accomplish the same task is to add a term $\exp\{i\int d^4x(i\varepsilon)\phi^2\}$ with $\varepsilon > 0$ in the generating functional. This will provide a strong damping to the Gaussian integral. The generating functional will then be well defined. The Green's function for the corresponding equation of motion $(\Box + \mu^2 - i\varepsilon)\phi = J$ is of the form

$$(\Box_x + \mu^2 - i\varepsilon)\Delta_F(x-y) = -\delta^4(x-y) \tag{1.168}$$

with the solution as given in eqn (1.166).

### Solution to Problem 1.13

**(a)** The minimization condition of the action (modified according to the $i\varepsilon$ prescription as discussed in the *Remark*):

$$S_J[\phi] = \int d^4x\,\mathcal{L}_J = \int d^4x\left[\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}(\mu^2 - i\varepsilon)\phi^2 + J\phi\right] \tag{1.169}$$

is simply the Euler–Lagrangian equation

$$\delta_\phi S_J = \frac{\partial\mathcal{L}_J}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}_J}{\partial(\partial_\mu\phi)}\right) = -(\mu^2 - i\varepsilon)\phi + J - \Box\phi = 0. \tag{1.170}$$

Thus the equation of motion is

$$(\Box_x + \mu^2 - i\varepsilon)\phi(x) = J(x). \tag{1.171}$$

This equation can be solved by the usual Green's function method

$$\phi_c(x) = -\int \Delta_F(x-y)J(y)\,d^4y \tag{1.172}$$

with the Green's function (the propagator) being defined by the equation

$$(\Box_x + \mu^2 - i\varepsilon)\Delta_F(x-y) = -\delta^4(x-y). \tag{1.173}$$

Equation (1.173) can be solved by Fourier transform with the solution being given by eqn (1.166).

**(b)** The change of variable $\phi(x) = \phi_c(x) + \eta(x)$ in the Lagrangian density of eqn (1.169)

$$S_J[\phi] = \int d^4x\, \mathcal{L}_J = \int d^4x \left[\tfrac{1}{2}\phi(-\Box - \mu^2)\phi + J\phi\right] \tag{1.174}$$

leads to an expression of the action in the form of

$$S_J[\phi_c + \eta] = \int d^4x \Big\{ \left[-\tfrac{1}{2}\phi_c(\Box + \mu^2)\phi_c + J\phi_c\right] \\ -\eta[(\Box + \mu^2)\phi_c - J] - \tfrac{1}{2}\eta[(\Box + \mu^2)\eta]\Big\}. \tag{1.175}$$

Because $\phi_c$ satisfies the equation of motion, the second term on the right-hand side vanishes and the first term can be simplified,

$$\begin{aligned}\int d^4x \left[-\tfrac{1}{2}\phi_c(\Box + \mu^2)\phi_c + J\phi_c\right] &= \int d^4x \left[-\tfrac{1}{2}\phi_c J + J\phi_c\right] \\ &= \tfrac{1}{2}\int d^4x\, J(x)\phi_c(x) \\ &= -\tfrac{1}{2}\int d^4x\, J(x)\Delta_F(x-y)J(y)\, d^4y.\end{aligned} \tag{1.176}$$

The generating functional being

$$W_o[J] = \int [d\phi] \exp\{iS_J[\phi]\} = \int [d\eta] \exp\{iS_J[\phi_c + \eta]\}, \tag{1.177}$$

we can then factor out the part of action which is independent of the $\eta(x)$ field:

$$\begin{aligned}W_o[J] &= \exp\left\{-\frac{i}{2}\int d^4x\, J(x)\Delta_F(x-y)J(y)\, d^4y\right\} \\ &\quad\times \int [d\eta(x)] \exp\left\{-\frac{i}{2}\int d^4x\, \eta(x)(\Box + \mu^2)\eta(x)\right\} \\ &= N \exp\left\{-\frac{i}{2}\int d^4x\, J(x)\Delta_F(x-y)J(y)\, d^4y\right\}\end{aligned} \tag{1.178}$$

where

$$N = \int [d\eta(x)] \exp\left\{-\frac{i}{2}\int d^4x\, \eta(x)(\Box + \mu^2)\eta(x)\right\} \tag{1.179}$$

is independent of $J(x)$. This is the desired result.

## 1.14 Poles in Green's function

Consider the following generalization of the *LSZ reduction formula*: let $T(q, \ldots)$ be a Fourier transform of the vacuum expectation value of a time-ordered product

$$T(q, \ldots) = \int d^4x\, e^{iq\cdot x}\, \langle 0|T(A(x)B(y)\cdots)|0\rangle\,, \tag{1.180}$$

where the operators $A(x)B(y)\cdots$ can be either elementary field operators like $\phi(x)$ or composite operators like $\phi^2(x)$. Suppose the operator $A(x)$ has a non-zero matrix element between the vacuum and some one-particle state carrying quantum number $a$,

$$\langle 0|A(0)|\mathbf{p}, a\rangle \neq 0, \qquad \text{with} \quad (\mathbf{p}^2 + m_a^2)^{1/2} = E_a. \tag{1.181}$$

Show that in the upper half of the energy plane $q_o > 0$ the function $T(q, \ldots)$ has a pole structure of

$$\lim_{q^2 \to m_a^2} T(q, \ldots) = i\,\frac{\langle 0|A(0)|\mathbf{p}, a\rangle\langle \mathbf{p}, a\,|T(B(y)\cdots)|0\rangle}{q^2 - m_a^2 + i\varepsilon} \quad \text{for } q_o > 0. \tag{1.182}$$

**Solution to Problem 1.14**

We only need to consider in detail the simplest non-trivial case of two operators. Generalization to cases involving more than two operators will be straightforward.

$$\begin{aligned}
T(q) &= \int d^4x\, e^{iq\cdot x}\, \langle 0|T(A(x)B(0))|0\rangle \\
&= \int d^4x\, e^{iq\cdot x}\, \{\theta(x)\langle 0|A(x)B(0)|0\rangle + \theta(-x)\langle 0|B(0)A(x)|0\rangle\} \\
&= \int d^4x\, e^{iq\cdot x} \Bigg\{\theta(x)\sum_n \langle 0|A(x)|n\rangle\langle n|B(0)|0\rangle \\
&\qquad + \theta(-x)\sum_k \langle 0|B(0)|k\rangle\langle k|A(x)|0\rangle\Bigg\}.
\end{aligned} \tag{1.183}$$

For the one-particle state $|\mathbf{p}, a\rangle$ in $\{|n\rangle\}$

$$\sum_n \to \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_a}, \tag{1.184}$$

we can use the translational invariance to write $\langle 0|A(x)|\mathbf{p}, a\rangle = \langle 0|A(0)|\mathbf{p}, a\rangle\, e^{-ip\cdot x}$, and

$$\begin{aligned}
\int d^4x\, e^{i(q-p)\cdot x}\theta(x) &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \int_0^\infty dt\, \exp\{i(q_0 - E_a)\cdot t\} \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \frac{i}{q_0 - E_a + i\varepsilon} \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \frac{i(q_0 + E_a)}{q^2 - m_a^2 + i\varepsilon},
\end{aligned} \tag{1.185}$$

where we have used $\mathbf{p} = \mathbf{q}$ so that

$$q_0 - E_a = \frac{q_0^2 - E_a^2}{(q_0 + E_a)} = \frac{q_0^2 - \mathbf{p}^2 - m_a^2}{(q_0 + E_a)} = \frac{q^2 - m_a^2}{(q_0 + E_a)}. \tag{1.186}$$

Inserting this into eqn (1.183), we can evaluate the function $T(q)$ in the limit of $q^2 \to m_a^2$ with $q_o > 0$

$$\begin{aligned} T(q) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_a}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q})\frac{2iE_a}{q^2 - m_a^2 + i\varepsilon} \\ &\quad \times \langle 0|A(0)|\mathbf{p}, a\rangle\langle \mathbf{p}, a|B(0)|0\rangle + \cdots \\ &= i\frac{\langle 0|A(0)|\mathbf{q}, a\rangle\langle \mathbf{q}, a|B(0)|0\rangle}{q^2 - m_a^2 + i\varepsilon} + \cdots \end{aligned} \tag{1.187}$$

where the ellipsis stands for the remaining terms, which are clearly free of poles and hence can be dropped in this limit. This is the desired result.

We also note that for the case of $\langle \mathbf{q}, a|A(0)|0\rangle \neq 0$ there will be a pole term in the $q_0 < 0$ region of the form

$$\lim_{q^2 \to m_a^2} T(q) = -i\frac{\langle 0|B(0)|\mathbf{q}, a\rangle\langle \mathbf{q}, a|A(0)|0\rangle}{q^2 - m_a^2 + i\varepsilon} \qquad \text{for } q_0 < 0. \tag{1.188}$$

# 2 Renormalization

## 2.1 Counterterms in $\lambda\phi^4$ theory and in QED

**(a)** Use the power-counting argument to construct counterterms and draw all the one-loop divergent 1PI graphs for the real scalar field theory with an interaction of

$$\mathcal{L}_{int} = -\frac{\lambda_1}{3!}\phi^3 - \frac{\lambda_2}{4!}\phi^4. \tag{2.1}$$

**(b)** Use the power-counting argument to construct counterterms for the QED Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

**Remark.** One of the key features of the QED theory is that it is invariant under the *gauge transformation* (see CL-Section 8.1 for details)

$$\begin{aligned}
\psi(x) &\to \psi'(x) = e^{-i\alpha(x)}\,\psi(x)\\
\bar{\psi}(x) &\to \bar{\psi}'(x) = e^{i\alpha(x)}\,\bar{\psi}(x)\\
A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\,\partial_\mu\alpha(x).
\end{aligned} \tag{2.3}$$

The desired counterterms must also be gauge invariant.

### Solution to Problem 2.1

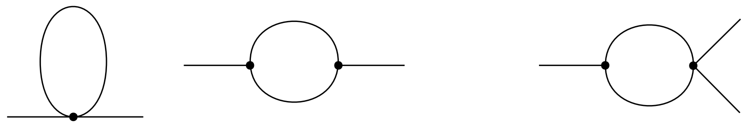
**(a)** The superficial degree of divergence $D$ is related to the number of external boson lines $B$ and the number of $\phi^3$ vertices $n_1$ by CL-eqn (2.133):

$$D = 4 - B - n_1. \tag{2.4}$$

**(i)** $B = 2$ (the *self-energy diagram*): see Fig. 2.1(a) for the one-loop divergent 1PI graph. Thus $D = 2 - n_1$. Since the number of external lines is even, $n_1$ must also be even: $n_1 = 0$ and 2, leading to quadratically divergent $\phi^2$ and logarithmically divergent $\partial_\mu\phi\,\partial^\mu\phi$ counterterms.

**(ii)** $B = 3$ (the $\phi^3$-*vertex diagram*): see Fig. 2.1(b) for the one-loop divergent 1PI graph. Thus $D = 1 - n_1 = 0$, as $n_1$ must be odd (hence $n_1 = 1$), leading to a logarithmically divergent $\phi^3$ counterterm.

**(iii)** $B = 4$ (the $\phi^4$-*vertex diagram*): see Fig. 2.2 for the one-loop divergent 1PI graph. Thus $D = 0$ leading to a logarithmically divergent $\phi^4$ counterterm.



(a) (b)

FIG. 2.1.

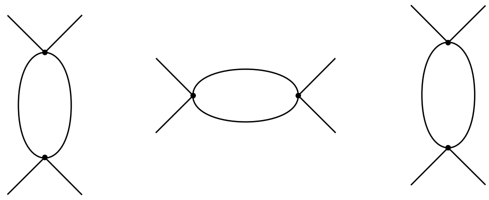
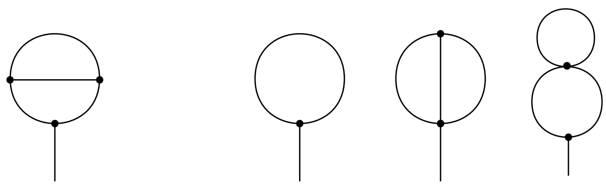


FIG. 2.2.



(a) (b)

FIG. 2.3.

**Remark.** We may also consider the 'tadpole diagrams' of Fig. 2.3 with $B = 1$ and $D = 3 - n_1$ with odd $n_1 = 1$ and 3. The quadratically divergent ones are shown in Fig. 2.3(b), while the logarithmic divergent one is shown in Fig. 2.3(a).

**(b)** Here both the external boson and fermion line numbers enter into CL-eqn (2.133)

$$D = 4 - B - \frac{3}{2}F. \tag{2.5}$$

Let us enumerate all possible terms starting with the lowest possible external fermion (electron) and boson (photon) lines:

**(i)** $F = 0$, $B = 2$ (the *vacuum polarization diagram*, Fig. 2.4(a)): thus the degree of divergence $D = 2$ (i.e. quadratically divergent). In order to have a finite term we need to expand this contribution $\pi_{\mu\nu}(k)$ beyond the second order in photon momentum $k$:

$$\pi_{\mu\nu}(k) = \pi_{\mu\nu}(0) + k^2 g_{\mu\nu}\pi_1(0) + k_\mu k_\nu \pi_2(0) + \widetilde{\pi}_{\mu\nu}(k). \tag{2.6}$$

Thus the required counterterms are $(A)^2$ and $(\partial A)^2$. But there is no gauge-invariant counterterm of the non-derivative form $(A)^2$. However, there is one gauge-invariant term of $(\partial A)^2$: $(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$, which is the same form as the photon kinetic energy term $F_{\mu\nu}F^{\mu\nu}$.

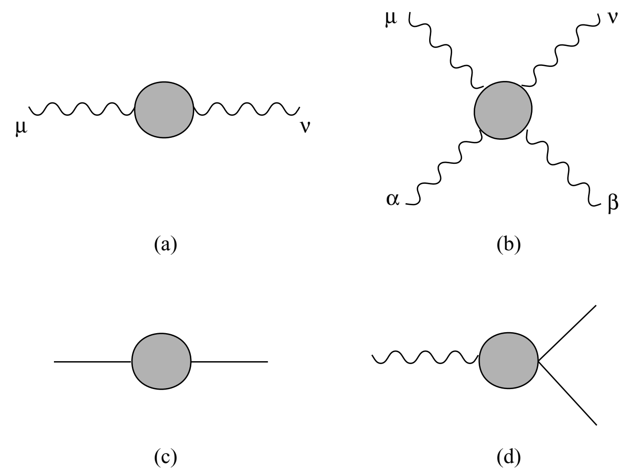



FIG. 2.4.

**(ii)** $F = 0$, $B = 4$ (the *photon–photon scattering diagram*, Fig. 2.4(b)): here we have $D = 0$ (i.e. logarithmically divergent).

$$\Gamma_{\mu\nu\lambda\rho}(k_i) = \Gamma_{\mu\nu\lambda\rho}(0) + \widetilde{\Gamma}_{\mu\nu\lambda\rho}(k_i) \tag{2.7}$$

where $\widetilde{\Gamma}_{\mu\nu\lambda\rho}(k_i)$ is convergent. The required counterterm is of the form $(A)^4$. However, it is not possible to construct such a term which is gauge invariant. Thus we would expect $\Gamma_{\mu\nu\lambda\rho}(k_i)$ itself to be convergent.

**(iii)** $F = 2$, $B = 0$ (the *electron self-energy diagram,* Fig. 2.4(c): the degree of divergence is one; hence it is linearly divergent.

$$\Sigma(p) = \Sigma(0) + \not{p}\,\Sigma'(0) + \widetilde{\Sigma}(p) \tag{2.8}$$

where we expect $\Sigma(0)$ to be linearly (or logarithmically) and $\Sigma'(0)$ to be logarithmically divergent, $\widetilde{\Sigma}(p)$ being convergent. The required counterterms are $\Sigma(0)\bar{\psi}\psi$ and $\Sigma'(0)\bar{\psi}\gamma^\mu\partial_\mu\psi$, respectively.

**(iv)** $F = 2$, $B = 1$ (the *electron–photon vertex diagram*, Fig. 2.4(d): it is logarithmically divergent, because $D = 0$.

$$\Gamma_\mu(p, q) = \Gamma_\mu(0) + \widetilde{\Gamma}_\mu(p, q) \tag{2.9}$$

with a counterterm of the form $\bar{\psi}\gamma^\mu\psi$.

## 2.2 Divergences in non-linear chiral theory

The non-linear $SU(2) \times SU(2)$ chiral Lagrangian is of the form

$$\mathcal{L} = \frac{f^2}{4}\,Tr(\partial^\mu U^\dagger\,\partial_\mu U) \tag{2.10}$$

where

$$U = \exp\left(\frac{i\,\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right),$$

and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices.

Also $f = 94\,\text{MeV}$ is the pion decay constant. (For more discussion of non-linear chiral theories, see Problems 5.7, 5.8, and 5.9.) Use power counting to enumerate all the superficially divergent Green's functions at the one-loop level and construct the appropriate counterterms.

**Solution to Problem 2.2**

The superficial degree of divergence is given by

$$D = 4 - B + \sum_i n_i(b_i + d_i - 4). \tag{2.11}$$

For the chiral Lagrangian, we have the number of derivatives in the $i$th vertex being $d_i = 2$, for all vertices. Substituting the topological relation

$$2(IB) + B = \sum_i n_i b_i \tag{2.12}$$

into

$$L = (IB) - \sum_i n_i + 1 \tag{2.13}$$

we get

$$L = \frac{1}{2}\sum_i n_i(b_i - 2) - \frac{B}{2} + 1. \tag{2.14}$$

The superficial degree of divergence is then

$$D = 4 - B + \sum_i n_i(b_i - 2) = 2 + 2L. \tag{2.15}$$

This gives the result that at the one-loop level ($L = 1$) the degree of divergence $D = 4$, independent of the external lines. It implies that the number of derivatives in the counterterms should be four or less. The term with two derivatives is just the term in the original Lagrangian. The four derivative terms should have the form

$$(\partial_\mu\phi)(\partial^\mu\phi)(\partial_\nu\phi)(\partial^\nu\phi)\phi^n, \qquad n \text{ even.} \tag{2.16}$$

Similarly for the other counterterms. Taking into account the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry, the counterterms are of the form,

$$\begin{aligned}
&[Tr(\partial^\mu U^\dagger\,\partial_\mu U)]^2\\
&Tr(\partial^\mu U^\dagger\,\partial_\mu U\,\partial^\nu U^\dagger\,\partial_\nu U)\\
&Tr(\partial^\mu U U^\dagger\,\partial_\nu U U^\dagger\,\partial_\mu U U^\dagger\,\partial^\nu U U^\dagger)\\
&Tr(\partial^\mu U^\dagger\,\partial^\nu U)Tr(\partial_\nu U^\dagger\,\partial_\mu U)\\
&Tr(\partial^\mu U U^\dagger\,\partial_\nu U U^\dagger)Tr(\partial_\mu U U^\dagger\,\partial^\nu U U^\dagger).
\end{aligned} \tag{2.17}$$

## 2.3 Divergences in lower-dimensional field theories

Consider theories with scalar field $\phi$, fermion field $\psi$, and massless vector gauge field $A_\mu$ in a $d$-dimensional space-time ($d-1$ space coordinates and 1 time coordinate).

Express the superficial degree of divergence of a diagram $D$ in terms of the number of external fermion $F$, boson $B$ lines, and the number of fermion $f_i$, boson $b_i$ lines and the number of derivatives $d_i$ in the $i$th type vertex. Namely, deduce the generalization of CL-eqn (2.133) to a $d$-dimensional field theory. Keep in mind that the propagator in $d$-dimensional field theory has exactly the same form as those in our physical four-dimensional momentum space. For example, the propagator for the gauge field $A_\mu$ in the $\xi = 1$ Feynman gauge is

$$i\Delta_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}. \tag{2.18}$$

Use the formula deduced in (a) to write down all possible renormalizable and super-renormalizable interactions for dimensions: *(i)* $d = 2$ and *(ii)* $d = 3$.

**Solution to Problem 2.3**

From the structure of the graph we have the relations

$$B + 2(IB) = \sum_i n_i b_i, \qquad F + 2(IF) = \sum_i n_i f_i, \tag{2.19}$$

as well as

$$(IB) + (IF) - \sum_i n_i + 1 = L \tag{2.20}$$

where $L$ is the number of loops in the graph. The superficial degree of divergence can be calculated by using the relation (2.20)

$$\begin{aligned} D &= dL - 2(IB) - (IF) + \sum_i n_i d_i \\ &= d\left((IB) + (IF) - \sum_i n_i + 1\right) - 2(IB) - (IF) + \sum_i n_i d_i. \end{aligned} \tag{2.21}$$

Eliminating *IB* and *IF* by eqn (2.19), we get

$$\begin{aligned} D &= \left(\frac{d-2}{2}\right)\left[\sum_i n_i b_i - B\right] + \left(\frac{d-1}{2}\right)\left[\sum_i n_i f_i - F\right] \\ &\quad + \sum_i n_i (d_i - d) + d \\ &= d - \left(\frac{d-2}{2}\right) B - \left(\frac{d-1}{2}\right) F \\ &\quad + \sum_i n_i \left[d_i + \left(\frac{d-2}{2}\right) b_i + \left(\frac{d-1}{2}\right) f_i - d\right]. \end{aligned}$$

Or

$$D = d - \left(\frac{d-2}{2}\right) B - \left(\frac{d-1}{2}\right) F + \sum_i n_i \delta_i \tag{2.22}$$

where $\delta_i$ is the index of divergence,

$$\delta_i = \left(\frac{d-2}{2}\right) b_i + \left(\frac{d-1}{2}\right) f_i + d_i - d. \tag{2.23}$$

Clearly, these results check with CL-eqn (2.133) for the case of dimension $d = 4$.

**(i) The $d = 2$ case:** Equations (2.22) and (2.23) are reduced to

$$D = 2 - \frac{F}{2} + \sum_i n_i \delta_i, \qquad \delta_i = \frac{f_i}{2} + d_i - 2. \tag{2.24}$$

- super-renormalizable interaction, $\delta_i < 0$

$$\begin{aligned} &\phi^n, \qquad n = 3, 4, \ldots \qquad \delta(\phi^n) = -2, \\ &\bar{\psi}\psi\phi^n, \qquad n = 1, 2, \ldots \qquad \delta(\bar{\psi}\psi\phi^n) = -1, \\ &\bar{\psi}\gamma_\mu\psi A^\mu, \qquad \delta(\bar{\psi}\gamma_\mu\psi A^\mu) = -1. \end{aligned} \tag{2.25}$$

  Note the interaction of the type $A_\mu A^\mu$ is super-renormalizable but not gauge invariance.
- renormalizable interaction, $\delta_i = 0$

$$\begin{aligned} d_i &= 0, \qquad (\bar{\psi}\psi)^2, \qquad (\bar{\psi}\psi)^2\phi^n \\ d_i &= 1, \qquad \bar{\psi}\gamma_\mu\psi\partial_\mu\phi\phi^n, \qquad \bar{\psi}\gamma_\nu\gamma_\mu\psi F^{\mu\nu}, \\ d_i &= 2, \qquad (\partial_\mu\phi\partial^\mu\phi)\phi^n. \end{aligned} \tag{2.26}$$

**(ii) The $d = 3$ case:** Equations (2.22) and (2.23) are reduced to

$$D = 3 - \frac{B}{2} - F + \sum_i n_i \delta_i, \qquad \delta_i = d_i + \frac{b_i}{2} + f_i - 3. \tag{2.27}$$

- super-renormalizable interaction, $\delta_i < 0$

$$\begin{aligned} d_i &= 0, \qquad \phi^3, \quad \phi^4, \quad \phi^5, \quad \bar{\psi}\gamma_\mu\psi A^\mu, \\ d_i &= 1, \qquad \partial_\mu\phi A^\mu\phi. \end{aligned} \tag{2.28}$$

- renormalizable interaction, $\delta_i = 0$

$$\begin{aligned} d_i &= 0, \quad \phi^6, \quad \bar{\psi}\psi\phi^2, \\ d_i &= 1, \qquad \partial_\mu\phi A^\mu\phi^2. \end{aligned} \tag{2.29}$$

## 2.4 $n$-Dimensional 'spherical' coordinates

In an $n$-dimensional space, the Cartesian coordinates can be parametrized in terms of the 'spherical' coordinates as

$$\begin{aligned}
x_1 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_2 \sin\theta_1, \\
x_2 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_2 \cos\theta_1, \\
x_3 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_3 \cos\theta_2, \\
\vdots & \quad \vdots \\
x_n &= r_n \cos\theta_{n-1}
\end{aligned} \tag{2.30}$$

where

$$0 \le \theta_1 \le 2\pi, \quad 0 \le \theta_2, \theta_3, \ldots, \theta_{n-1} \le \pi, \tag{2.31}$$

and

$$r_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \tag{2.32}$$

Show that the $n$-dimensional infinitesimal volume is given by

$$\begin{aligned}
dx_1\, dx_2\, dx_3 \ldots dx_n = r_n^{n-1} (\sin\theta_{n-1})^{n-2} (\sin\theta_{n-2})^{n-3} \cdots \\
\times (\sin\theta_2)(d\theta_1\, d\theta_2 \cdots d\theta_{n-1})\, dr_n
\end{aligned} \tag{2.33}$$

as used in CL-p. 53.

**Solution to Problem 2.4**

We will solve this problem by finding the relation between the volume factors in $n$ and $n-1$ dimensions. Namely, we will proceed from the simplest $n = 2$ to higher-dimensional cases:

$$(n = 2) \longrightarrow (n = 3) \longrightarrow (n = 4) \longrightarrow (\text{general } n). \tag{2.34}$$

**(a)** $n = 2$
Here the two Cartesian coordinates $(x_1, x_2)$ are related to the familiar polar coordinates $(r_2, \theta_1)$ by

$$x_1 = r_2 \sin\theta_1, \quad x_2 = r_2 \cos\theta_1, \qquad 0 \le \theta_1 \le 2\pi, \quad r_2^2 = x_1^2 + x_2^2. \tag{2.35}$$

The distance $ds_2$ between neighbouring points is given by

$$(ds_2)^2 = (dx_1)^2 + (dx_2)^2 = (dr_2)^2 + r_2^2 (d\theta_1)^2. \tag{2.36}$$

The 'volume' element is then the product of segments in orthogonal directions, i.e. the product of the coordinate differential with the appropriate coefficients as indicated by the (quadratic) distance relation:

$$dV_2 = dx_1\, dx_2 = (dr_2)(r_2 d\theta_1) = r_2 dr_2\, d\theta_1. \tag{2.37}$$

**(b)** $n = 3$

Consider a sphere in three dimensions. If we cut this sphere by a plane perpendicular to the $x_3$-axis, we get a series of circles in the planes spanned by Cartesian coordinates $(x_1, x_2)$ which are related to the polar coordinates $(r_2, \theta_1)$ by

$$x_1 = r_2 \sin\theta_1, \quad x_2 = r_2 \cos\theta_1, \qquad 0 \le \theta_1 \le 2\pi, \quad r_2^2 = x_1^2 + x_2^2. \tag{2.38}$$

The infinitesimal distance on this plane can be expressed in these two coordinate systems as

$$(dx_1)^2 + (dx_2)^2 = (dr_2)^2 + r_2^2(d\theta_1)^2. \tag{2.39}$$

We can also cut the sphere by a plane containing the $x_3$-axis, resulting in a series of 'vertical circles'. On these two-dimensional subspaces, the Cartesian coordinates are $(r_2, x_3)$ and the corresponding polar coordinates are $(r_{3,}\theta_2)$. We recognize that $\theta_2$ is the usual polar angle.

$$r_2 = r_3 \sin\theta_2, \quad x_3 = r_3 \cos\theta_2, \qquad 0 \le \theta_2 \le \pi. \tag{2.40}$$

$$r_3^2 = r_2^2 + x_2^2 = x_1^2 + x_2^2 + x_3^2. \tag{2.41}$$

As in eqn (2.39), the infinitesimal distance can be expressed in two equivalent ways:

$$(dr_2)^2 + (dx_3)^2 = (dr_3)^2 + r_3^2(d\theta_2)^2. \tag{2.42}$$

Combining the two sets of coordinates in eqns (2.38) and (2.40), we get the usual spherical coordinate relations,

$$\begin{aligned} x_1 &= r_3 \sin\theta_2 \sin\theta_1, \\ x_2 &= r_3 \sin\theta_2 \cos\theta_1, \\ x_3 &= r_3 \cos\theta_2. \end{aligned} \tag{2.43}$$

We can turn the distance formula

$$(ds_3)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

into

$$(ds_3)^2 = (dr_2)^2 + r_2^2(d\theta_1)^2 + (dx_3)^2 \tag{2.44}$$

by using eqn (2.39). This can be further reduced, by using eqns (2.40) and (2.42), to

$$\begin{aligned} (ds_3)^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 \\ &= (dr_3)^2 + r_2^2(d\theta_1)^2 + r_3^2(d\theta_2)^2 \\ &= (dr_3)^2 + r_3^2 \sin\theta_2^2(d\theta_1)^2 + r_3^2(d\theta_2)^2. \end{aligned} \tag{2.45}$$

The volume element can be obtained from the product of these three terms

$$dV_3 = (dr_3)(r_3 \sin\theta_2\, d\theta_1)(r_3\, d\theta_2) = r_3^2 \sin\theta_2(dr_3\, d\theta_1\, d\theta_2). \tag{2.46}$$

**(c)** $n = 4$
We can also imagine cutting the sphere in four dimensions by the manifold ($x_4 =$ constant) to get a three-sphere with radius $r_3 = r_4 \sin\theta_3$, where we can introduce three-dimensional spherical coordinates $(r_3, \theta_1, \theta_2)$ as in eqn (2.43)

$$\begin{aligned}
x_1 &= r_3 \sin\theta_2 \sin\theta_1 = (r_4 \sin\theta_3) \sin\theta_2 \sin\theta_1, \\
x_2 &= r_3 \sin\theta_2 \cos\theta_1 = (r_4 \sin\theta_3) \sin\theta_2 \cos\theta_1, \\
x_3 &= r_3 \cos\theta_2 = (r_4 \sin\theta_3) \cos\theta_2,
\end{aligned} \tag{2.47}$$

with the distance formula (2.45)

$$(dx_1)^2 + (dx_2)^2 + (dx_3)^2 = (dr_3)^2 + r_3^2(d\theta_2)^2 + r_3^2 \sin^2\theta_2(d\theta_1)^2. \tag{2.48}$$

Now we introduce two-dimensional polar coordinates $(r_4, \theta_3)$ in the $(r_3, x_4)$ plane

$$r_3 = r_4 \sin\theta_3, \qquad x_4 = r_4 \cos\theta_3, \tag{2.49}$$

with the distance formula

$$(dr_3)^2 + (dx_4)^2 = (dr_4)^2 + r_4^2(d\theta_3)^2. \tag{2.50}$$

In this way the infinitesimal distance in this four-dimensional space can be rewritten by using eqns (2.48) and (2.50)

$$\begin{aligned}
(ds_4)^2 &= [(dx_1)^2 + (dx_2)^2 + (dx_3)^2] + (dx_4)^2 \\
&= [(dr_3)^2 + r_3^2(d\theta_2)^2 + r_3^2 \sin^2\theta_2(d\theta_1)^2] + (dx_4)^2 \\
&= [(dr_3)^2 + (dx_4)^2] + r_3^2(d\theta_2)^2 + r_3^2 \sin^2\theta_2(d\theta_1)^2 \\
&= (dr_4)^2 + r_4^2(d\theta_3)^2 + r_3^2(d\theta_2)^2 + r_3^2 \sin^2\theta_2(d\theta_1)^2.
\end{aligned} \tag{2.51}$$

The infinitesimal volume element is then

$$\begin{aligned}
dv_4 &= (dr_4)(r_4\, d\theta_3)(r_3\, d\theta_2)(r_3 \sin\theta_2\, d\theta_1) \\
&= r_4^3 \sin^2\theta_3 \sin\theta_2(dr_4\, d\theta_1\, d\theta_2\, d\theta_3)
\end{aligned} \tag{2.52}$$

where we have used eqn (2.49) to reach the last expression.

**(d)** General $n$

$$\begin{aligned}
x_1 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_2 \sin\theta_1, \\
x_2 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_2 \cos\theta_1, \\
x_3 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_3 \cos\theta_2, \\
&\vdots \qquad \vdots \\
x_n &= r_n \cos\theta_{n-1}
\end{aligned} \tag{2.53}$$

where

$$r_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \quad 0 \le \theta_1 \le 2\pi, \quad 0 \le \theta_2, \theta_3, \ldots, \theta_{n-1} \le \pi. \tag{2.54}$$

The infinitesimal distance is

$$
\begin{aligned}
(ds_n)^2 &= (dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2 \\
&= (dr_n)^2 + r_n^2 (d\theta_{n-1})^2 + r_n^2 \sin^2\theta_{n-1}(d\theta_{n-2})^2 \\
&\quad + r_n^2 \sin^2\theta_{n-1}\sin^2\theta_{n-2}(d\theta_{n-3})^2 + \cdots \\
&\quad + r_n^2 \sin^2\theta_{n-1}\sin^2\theta_{n-2}\cdots\sin^2\theta_2(d\theta_1)^2
\end{aligned} \tag{2.55}
$$

and the volume element is

$$
\begin{aligned}
dV_n &= dx_1\, dx_2 \cdots dx_n \\
&= (r_n)^{n-1}(\sin\theta_{n-1})^{n-2}(\sin\theta_{n-2})^{n-3}\cdots \\
&\quad \times \sin\theta_2 (dr_n\, d\theta_1\, d\theta_2 \cdots d\theta_{n-1}).
\end{aligned} \tag{2.56}
$$

## 2.5 Some integrals in dimensional regularization

Use the dimensional regularization to derive the following results for the Feynman integrals with denominator power $\alpha$ in $n$ dimensions:

**(a)**

$$
\begin{aligned}
I_0(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\
&= i\frac{(-\pi)^{n/2}}{(2\pi)^n}\frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)}\frac{1}{(M^2 - p^2 + i\varepsilon)^{\alpha-(n/2)}}.
\end{aligned} \tag{2.57}
$$

**(b)**

$$
\begin{aligned}
I_\mu(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\
&= -p_\mu I_0(\alpha, n).
\end{aligned} \tag{2.58}
$$

**(c)**

$$
\begin{aligned}
I_{\mu\nu}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\
&= I_0(\alpha, n)\left[p_\mu p_\nu + \frac{1}{2} g_{\mu\nu}\frac{M^2 - p^2}{(\alpha - n/2 - 1)}\right].
\end{aligned} \tag{2.59}
$$

**(d)**

$$
\begin{aligned}
I_{\mu\nu\rho}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu k_\rho}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\
&= I_0(\alpha, n)\Big[p_\mu p_\nu p_\rho + \tfrac{1}{2}(g_{\mu\nu}p_\rho + g_{\mu\rho}p_\nu + g_{\nu\rho}p_\mu) \\
&\qquad \times \frac{M^2 - p^2}{(\alpha - n/2 - 1)}\Big].
\end{aligned}
$$

**Solution to Problem 2.5**

**(a)** In the Feynman integral

$$I_0(\alpha, n) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^\alpha}, \tag{2.60}$$

the denominator can be written as

$$\begin{aligned} D &= k^2 + 2p \cdot k + M^2 + i\varepsilon = (p+k)^2 + (M^2 - p^2) + i\varepsilon \\ &= k'^2 + (M^2 - p^2) + i\varepsilon \end{aligned} \tag{2.61}$$

where $k' = k + p$. For the case $p^2 \geq M^2$, we can perform the Wick rotation to get

$$D = -\bar{k}^2 + M^2 - p^2 + i\varepsilon = -(\bar{k}^2 + a^2) \tag{2.62}$$

where $a^2 = p^2 - M^2 - i\epsilon$ and

$$I_0(\alpha, n) = i \int \frac{d^n k}{(2\pi)^n} \frac{1}{(-1)^\alpha} \frac{1}{(k^2 + a^2)^\alpha}. \tag{2.63}$$

As usual [see CL-eqn (2.112)], the $n$-dimensional angular integration gives

$$\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2.64}$$

Then

$$\begin{aligned} I_0(\alpha, n) &= \frac{i(-1)^{-\alpha}}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{k^{n-1} dk}{(k^2 + a^2)^\alpha} \\ &= \frac{i(-1)^{-\alpha}}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{2} \int_0^\infty \frac{t^{n/2-1} dt}{(t + a^2)^\alpha} \\ &= \frac{i(-1)^{-\alpha}}{(2\pi)^n} \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{1}{(a^2)^{\alpha-(n/2)}} \frac{\Gamma(n/2)\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \\ &= \frac{(-\pi)^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \frac{1}{(M^2 - p^2 + i\varepsilon)^{\alpha - n/2}}. \end{aligned} \tag{2.65}$$

One of the most common convergent Feynman integrals has $\alpha = 3$,

$$\begin{aligned} I_0(3, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^3} \\ &= \frac{(-\pi)^{n/2}}{(2\pi)^n} \frac{\Gamma(3 - (n/2))}{\Gamma(3)} \frac{1}{(M^2 - p^2 + i\varepsilon)^{3-n/2}} \end{aligned} \tag{2.66}$$

which gives, for $n = 4$,

$$\begin{aligned} I_0(3, 4) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^3} \\ &= \frac{i}{32\pi^2} \frac{1}{(M^2 - p^2 + i\varepsilon)}. \end{aligned} \tag{2.67}$$

**(b)**
$$I_\mu(\alpha, n) = \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha}. \tag{2.68}$$

As before, we set $k' = k + p$. Then

$$I_\mu(\alpha, n) = \int \frac{d^n k'}{(2\pi)^n} \frac{k'_\mu - p_\mu}{(k^2 + a^2)^\alpha}. \tag{2.69}$$

and the term linear in $k'_\mu$ gives zero because of the symmetric integration. The result is

$$I_\mu(\alpha, n) = -p_\mu I_0(\alpha, n). \tag{2.70}$$

**(c)**
$$\begin{aligned} I_{\mu\nu}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\ &= \int \frac{d^n k}{(2\pi)^n} \frac{(k'_\mu - p_\mu)(k'_\nu - p_\nu)}{(k^2 + a^2)^\alpha} \\ &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + a^2)^\alpha} + p_\mu p_\nu \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + a^2)^\alpha}. \end{aligned} \tag{2.71}$$

In the first term we can replace $k_\mu k_\nu$ by $(1/n)k^2 g_{\mu\nu}$ to get

$$\begin{aligned} \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + a^2)^\alpha} &= \frac{g_{\mu\nu}}{n} \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{(k^2 + a^2)^\alpha} \\ &= \frac{g_{\mu\nu}}{n} \int \frac{d^n k}{(2\pi)^n} \frac{k^2 + a^2 - a^2}{(k^2 + a^2)^\alpha} \\ &= \frac{g_{\mu\nu}}{n} \left[ I_0(\alpha - 1, n) - a^2 I_0(\alpha, n) \right]. \end{aligned} \tag{2.72}$$

Using the identity $\Gamma(x+1) = x\Gamma(x)$, we get

$$I_0(\alpha - 1, n) = \frac{(\alpha - 1)a^2}{(\alpha - 1 - n/2)} I_0(\alpha, n) \tag{2.73}$$

and

$$I_{\mu\nu}(\alpha, n) = \left[ p_\mu p_\nu + \tfrac{1}{2} g_{\mu\nu}[M^2 - p^2] \frac{1}{(\alpha - n/2 - 1)} \right] I_0(\alpha, n). \tag{2.74}$$

For the case $\alpha = 4$, we have

$$\begin{aligned} I_{\mu\nu}(4, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^4} \\ &= \left[ p_\mu p_\nu + \tfrac{1}{2} g_{\mu\nu}[M^2 - p^2] \frac{1}{(3 - n/2)} \right] I_0(4, n). \end{aligned} \tag{2.75}$$

which gives for $n = 4$,

$$I_{\mu\nu}(4,4) = \left[p_\mu p_\nu + \tfrac{1}{2} g_{\mu\nu}[M^2 - p^2]\right] \frac{i}{96\pi^2} \frac{1}{(M^2 - p^2 + i\varepsilon)}. \tag{2.76}$$

**(d)**

$$\begin{aligned} I_{\mu\nu\rho}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu k_\rho}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\ &= \int \frac{d^n k}{(2\pi)^n} \frac{(k'_\mu - p_\mu)(k'_\nu - p_\nu)(k'_\rho - p_\rho)}{(k^2 + a^2)^\alpha} \end{aligned} \tag{2.77}$$

Dropping terms with odd powers of $k'$, we get

$$\begin{aligned} I_{\mu\nu\rho}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + a^2)^\alpha} [-(k_\mu k_\nu p_\rho + k_\mu p_\nu k_\rho + p_\mu k_\nu k_\rho) - p_\mu p_\nu p_\rho] \\ &= -p_\mu p_\nu p_\rho I_0(\alpha, n) - (p_\rho I_{\mu\nu} + p_\mu I_{\nu\rho} + p_\nu I_{\mu\rho}) \\ &= I_0(\alpha, n)\left[4 p_\mu p_\nu p_\rho + \tfrac{1}{2}(g_{\mu\nu} p_\rho + g_{\mu\rho} p_\nu + g_{\nu\rho} p\mu) \frac{M^2 - p^2}{(\alpha - n/2 - 1)}\right]. \end{aligned}$$

## 2.6 Vacuum polarization and subtraction schemes

Use the dimensional regularization to compute the one-loop vacuum polarization in QED**.**

### Solution to Problem 2.6

The usual vacuum polarization in QED is given by

$$\begin{aligned} i\pi^{\alpha\beta}(q) &= (q^\alpha q^\beta - q^2 g^{\alpha\beta}) i\pi(q^2) \\ &= -(-ie_0)^2 \int \frac{d^4 p}{(2\pi)^4} Tr\left[\gamma^\alpha \frac{i}{\not p - m + i\varepsilon} \gamma^\beta \frac{i}{\not p - \not q - m + i\varepsilon}\right]. \end{aligned} \tag{2.78}$$

In the dimensional regularization, we can replace

$$\int \frac{d^4 p}{(2\pi)^4} \to \mu^\varepsilon \int \frac{d^d p}{(2\pi)^d} \tag{2.79}$$

where $\varepsilon = 4 - d$ and $\mu$ is some arbitrary mass scale that one can introduce in the dimensional regularization scheme. The integrand can be simplified as

$$\begin{aligned} &-Tr\left[\gamma^\alpha \frac{i}{\not p - m + i\varepsilon} \gamma^\beta \frac{i}{\not p - \not q - m + i\varepsilon}\right] \\ &\quad = -\left[\frac{1}{p^2 - m^2 + i\varepsilon}\right]\left[\frac{1}{(p-q)^2 - m^2 + i\varepsilon}\right] \\ &\quad\quad \times Tr[\gamma^\alpha(\not p + m)\gamma^\beta(\not p - \not q + m)]. \end{aligned} \tag{2.80}$$

The numerator is of the form

$$N^{\alpha\beta} = Tr[\gamma^\alpha \not{p} \gamma^\beta (\not{p} - \not{q})] + m^2 Tr(\gamma^\alpha \gamma^\beta). \tag{2.81}$$

Dirac algebra in $d$ dimensions gives

$$\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta} I_d \tag{2.82}$$

where $I_d$ is the identity matrix in $d$-dimensional Dirac algebra space, with the trace

$$TrI_d = f(d). \tag{2.83}$$

Here $f(d)$ can be any function so long as it has the property $f(4) = 4$. It is straightforward to see that

$$Tr(\not{a}\not{b}\not{c}\not{d}) = f(d)[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]. \tag{2.84}$$

Using these we get for the numerator

$$N^{\alpha\beta} = f(d)\left\{[p^\alpha(p-q)^\beta + p^\beta(p-q)^\alpha - g^{\alpha\beta} p\cdot(p-q)] + m^2 g^{\alpha\beta}\right\}. \tag{2.85}$$

The denominator is calculated in the usual way,

$$D = \frac{1}{(p^2 - m^2 + i\varepsilon)} \frac{1}{[(p-q)^2 - m^2 + i\varepsilon]} = \int_0^1 \frac{d\alpha}{A^2} \tag{2.86}$$

where

$$\begin{aligned} A &= (1-\alpha)(p^2 - m^2) + \alpha[(p-q)^2 - m^2] \\ &= p^2 - 2\alpha p\cdot q - m^2 + \alpha q^2 \\ &= (p - \alpha q)^2 - a^2 \end{aligned} \tag{2.87}$$

with

$$a^2 = m^2 - \alpha(1-\alpha)q^2. \tag{2.88}$$

The vacuum polarization is then of the form

$$\pi^{\alpha\beta}(q) = ie_0^2 \int d\alpha \int \frac{d^d p}{(2\pi)^d} \mu^\varepsilon \frac{N^{\alpha\beta}}{[(p-\alpha q)^2 - a^2]}. \tag{2.89}$$

To simplify the integration we shift the variable, $p \to p + \alpha q$. Then the numerator (2.85) becomes

$$\begin{aligned} N^{\alpha\beta} \to\ & f(d)\big\{[(p+\alpha q)^\alpha(p-(1-\alpha)q)^\beta + (p+\alpha q)^\beta(p-(1-\alpha)q)^\alpha \\ & - g^{\alpha\beta}[m^2 - (p+\alpha q)\cdot(p-(1-\alpha)q)]\big\} \\ =\ & f(d)\big\{2p^\alpha p^\beta - 2\alpha(1-\alpha)q^\alpha q^\beta \\ & + g^{\alpha\beta}[m^2 - (p^2 - \alpha(1-\alpha)q^2)]\big\} \end{aligned} \tag{2.90}$$

where we have dropped terms linear in $p$ which will vanish under the symmetric integration in $p$. Then we have

$$\pi^{\alpha\beta}(q) = ie_0^2\mu^\varepsilon f(d)\int_0^1 d\alpha \int \frac{d^d p}{(2\pi)^d}\frac{1}{(p^2-a^2)^2} \times \Big\{2p^\alpha p^\beta - 2\alpha(1-\alpha)q^\alpha q^\beta + g^{\alpha\beta}[m^2 - (p^2 - \alpha(1-\alpha)q^2)]\Big\}. \tag{2.91}$$

The term proportional to $g^{\alpha\beta}$ can be written as

$$-[p^2 - m^2 + \alpha(1-\alpha)q^2] + 2\alpha(1-\alpha)q^2 = -(p^2-a^2) + 2\alpha(1-\alpha)q^2$$

so we get

$$\begin{aligned}\pi^{\alpha\beta}(q) &= ie_0^2\mu^\varepsilon f(d)\int_0^1 d\alpha \int \frac{d^d p}{(2\pi)^d}\left\{\frac{2p^\alpha p^\beta}{(p^2-a^2)^2} - \frac{2\alpha(1-\alpha)q^\alpha q^\beta}{(p^2-a^2)^2} + \frac{2\alpha(1-\alpha)q^2 g^{\alpha\beta}}{(p^2-a^2)^2} - \frac{g^{\alpha\beta}}{p^2-a^2}\right\}\\ &= ie_0^2\mu^\varepsilon f(d)\int_0^1 d\alpha \int \frac{d^d p}{(2\pi)^d}\left\{\left[\frac{2p^\alpha p^\beta}{(p^2-a^2)^2} - \frac{g^{\alpha\beta}}{p^2-a^2}\right] + \frac{2\alpha(1-\alpha)(q^2 g^{\alpha\beta} - q^\alpha q^\beta)}{(p^2-a^2)^2}\right\}.\end{aligned} \tag{2.92}$$

The relevant formulae for the dimensional integration are eqns (2.57) and (2.59).

$$I_0 = \int \frac{d^d p}{(2\pi)^d}\frac{1}{(p^2-a^2)^\alpha} = \frac{i(-\pi)^{d/2}}{(2\pi)^d}\frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)}\frac{1}{(-a^2)^{\alpha-d/2}}. \tag{2.93}$$

$$I_{\mu\nu} = \int \frac{d^d p}{(2\pi)^d}\frac{p^\mu p^\nu}{(p^2-a^2)^\alpha} = \frac{g^{\mu\nu}(-a^2)}{(\alpha - d/2 - 1)} I_0. \tag{2.94}$$

Using these we get

$$\int \frac{d^d p}{(2\pi)^d}\frac{p^\alpha p^\beta}{(p^2-a^2)^2} = \frac{g^{\alpha\beta}}{(1-d/2)}\frac{i(-\pi)^{d/2}}{(2\pi)^d}\frac{\Gamma(2-d/2)}{\Gamma(2)}\frac{1}{(-a^2)^{1-d/2}}. \tag{2.95}$$

$$\int \frac{d^d p}{(2\pi)^d}\frac{g^{\alpha\beta}}{(p^2-a^2)} = g^{\alpha\beta}\frac{i(-\pi)^{d/2}}{(2\pi)^d}\frac{\Gamma(1-d/2)}{\Gamma(1)}\frac{1}{(-a^2)^{1-d/2}}. \tag{2.96}$$

Using the identity

$$\Gamma\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) \tag{2.97}$$

we can show that

$$\int \frac{d^d p}{(2\pi)^d}\left[\frac{2p^\alpha p^\beta}{(p^2-a^2)^2}-\frac{g^{\alpha\beta}}{p^2-a^2}\right]=0. \tag{2.98}$$

Now the vacuum polarization is of the gauge-invariant form,

$$\pi^{\alpha\beta}(q)=e_0^2\mu^\varepsilon f(d)\frac{1}{16\pi^2(4\pi)^{-\varepsilon/2}}(q^\alpha q^\beta-q^2g^{\alpha\beta})\int d\alpha\frac{2\alpha(1-\alpha)}{[m^2-\alpha(1-\alpha)q^2]^{\varepsilon/2}}.$$

Expanding in the power of $\varepsilon$,

$$\frac{\Gamma(\varepsilon/2)}{(4\pi)^{-\varepsilon/2}}=\frac{2}{\varepsilon}-\gamma+\ln(4\pi)+O(\varepsilon) \tag{2.99}$$

where $\gamma=0.5772\ldots$ is the Euler constant,

$$\mu^\varepsilon(a^2)^{-\varepsilon/2}=1-\frac{\varepsilon}{2}(\ln a^2-\ln\mu^2) \tag{2.100}$$

we get

$$\pi(q)=\frac{e_0^2f(d)}{8\pi^2}\int_0^1 d\alpha\,\alpha(1-\alpha)\left\{\frac{2}{\varepsilon}-\gamma+\ln(4\pi)-\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right]\right\}.$$

Write

$$f(d)=f(4)+(d-4)f''(4)+\cdots=4(1+a\varepsilon+\cdots) \tag{2.101}$$

where $a=-\frac{1}{2}f'(4)$. Then we have

$$\pi(q)=\frac{e_0^2}{2\pi^2}\left\{\frac{1}{3}\left[\frac{2}{\varepsilon}-\gamma+\ln(4\pi)+2a\right]\right.$$
$$\left.-\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right]\right\}.$$

**Different subtraction schemes**

From this we see that different choice of $f(d)$ corresponds to different constants in the finite part of $\pi(q)$, which is arbitrary anyway. For convenience, we can choose $a=0$, or $f(d)=4$ for all $d$.

**Minimal subtraction scheme (MS).** Here we subtract out the pole in $\varepsilon$ to get

$$\pi_{\mathrm{MS}}(q)=\frac{e_0^2}{2\pi^2}\left\{\frac{1}{3}(-\gamma+\ln 4\pi)\right.$$
$$\left.-\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right]\right\}.$$

This corresponds to choosing the renormalization constant,

$$(Z_2^{-1})_{\mathrm{MS}}=1+\frac{e_0^2}{3\varepsilon}. \tag{2.102}$$

**Modified minimal subtraction scheme ($\overline{\textbf{MS}}$).** The renormalization constant is chosen so that the term $(-\gamma + \ln 4\pi)$ is also removed from the finite part,

$$\pi_{\overline{\text{MS}}}(q) = \frac{-e_0^2}{2\pi^2}\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right]. \tag{2.103}$$

From this result we can study the low- and high-energy behaviour of $\pi(q)$. For $|q|^2 \ll m^2$,

$$\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right] \simeq \ln\frac{m^2}{\mu^2} - \alpha(1-\alpha)\frac{q^2}{m^2} \tag{2.104}$$

and

$$\begin{aligned}
&\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right]\\
&\quad= \int_0^1 d\alpha\,\alpha(1-\alpha)\left[\ln\frac{m^2}{\mu^2} - \alpha(1-\alpha)\frac{q^2}{m^2}\right]\\
&\quad= \frac{1}{6}\ln\frac{m^2}{\mu^2} - \frac{1}{30}\left(\frac{q^2}{m^2}\right) + \cdots .
\end{aligned} \tag{2.105}$$

Then we have

$$\pi_{\overline{\text{MS}}}(q) = -\frac{e_0^2}{12\pi^2}\ln\frac{m^2}{\mu^2} + \frac{e_0^2}{60\pi^2}\left(\frac{q^2}{m^2}\right) + \cdots . \tag{2.106}$$

In the other limit $|q|^2 \gg m^2$,

$$\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right] \simeq \ln\left(\frac{-\alpha(1-\alpha)q^2}{\mu^2}\right)\left[1 - \frac{m^2}{q^2\alpha(1-\alpha)} + \cdots\right]$$

and

$$\begin{aligned}
&\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2-\alpha(1-\alpha)q^2}{\mu^2}\right]\\
&\quad= \ln\left(\frac{-q^2}{\mu^2}\right)\int_0^1 d\alpha\,\alpha(1-\alpha) + \int_0^1 d\alpha\,\alpha(1-\alpha)\ln[\alpha(1-\alpha)]\\
&\quad= \frac{1}{6}\ln\left(\frac{-q^2}{\mu^2}\right) - \frac{5}{18} + \cdots .
\end{aligned} \tag{2.107}$$

Thus the limit is

$$\pi_{\overline{\text{MS}}}(q) = -\frac{e_0^2}{12\pi^2}\ln\left(\frac{-q^2}{\mu^2}\right) + \frac{5e_0^2}{36\pi^2} + \cdots . \tag{2.108}$$

## 2.7 Renormalization of $\lambda\phi^3$ theory in $n$ dimensions

Consider the $\lambda\phi^3$ theory where the Lagrangian is of the form,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\bar{\lambda}_0}{3!}\phi^3. \tag{2.109}$$

**(a)** Show that $\bar{\lambda}_0$ has dimension $(6-n)/2$ in $n$-dimensional space-time. To have $\lambda_0$ with fixed dimension for arbitrary $n$, we can define

$$\bar{\lambda}_0 = \lambda_0(\mu)^{(4-n)/2}, \qquad \bar{\lambda}_0 = \lambda_0 \mu^{\varepsilon}, \quad \text{with} \quad \varepsilon = \frac{4-n}{2} \tag{2.110}$$

where $\mu$ is an arbitrary mass scale.

**(b)** Show that the one-loop divergent graphs for the case $n = 4$ are those given in Fig. 2.5.

**(c)** Carry out the renormalization program for this theory by using the $\overline{\text{MS}}$ scheme.

### Solution to Problem 2.7

**(a)** Since the action $S = \int d^n x\, \mathcal{L}$ is dimensionless, $\mathcal{L}$ has the dimension $n$. From the mass term or the kinetic energy term we see that $\phi$ has dimension $(n-2)/2$, which gives the dimension of $\bar{\lambda}_0$ as $(6-n)/2$.

**(b)** From eqn (2.23) we know that the index of divergence for the $\phi^3$ interaction is

$$\delta = 3\left(\frac{n-2}{2}\right) - n = \left(\frac{n-6}{2}\right) \tag{2.111}$$

which, as expected, is just the negative of the dimension of $\bar{\lambda}_0$. The superficial degree of divergence is then

$$D = n - \left(\frac{n-2}{2}\right) B + \nu\delta \tag{2.112}$$

where $\nu$ is the number of $\phi^3$ interactions in the graph. The number of loops $L$ is given by eqn (2.14)

$$L = \tfrac{1}{2}(\nu - B) + 1. \tag{2.113}$$

Thus in one-loop we have $\nu = B$, and

$$D = n - 2B, \tag{2.114}$$

and for four-dimensional theories $n = 4$, only $B = 2$ self-energy and $B = 1$ tadpole graphs are divergent.

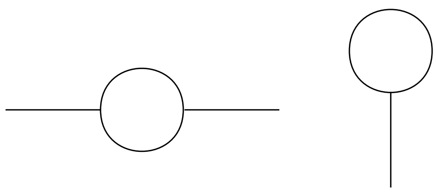


FIG. 2.5. Self-energy and tadpole diagrams in the $\lambda\phi^3$ theory.

**(c-I) The self-energy graph**

$$-i\Sigma(p) = \frac{(i\lambda_0\mu^\varepsilon)^2}{2}\int \frac{d^nk}{(2\pi)^n}\frac{i}{(k^2-m^2)}\frac{i}{[(k-p)^2-m^2]}. \tag{2.115}$$

It is straightforward to evaluate this integral to get

$$\Sigma(p) = \frac{\lambda_0^2\mu^{2\varepsilon}}{2}\frac{\Gamma(\varepsilon)}{(4\pi)^{n/2}}\int_0^1 \frac{d\alpha}{[m^2-\alpha(1-\alpha)p^2-i\varepsilon]^\varepsilon}. \tag{2.116}$$

Using

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon), \qquad a^\varepsilon = 1 + \varepsilon \ln a + O(\varepsilon^2) \tag{2.117}$$

we can expand $\Sigma(p)$ around $\varepsilon = 0$, to get

$$\Sigma(p) = \frac{\lambda_0^2}{2}\frac{1}{(4\pi)^2}\left\{\frac{1}{\varepsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right) - \int_0^1 d\alpha \ln\left[\frac{m^2-\alpha(1-\alpha)p^2-i\varepsilon}{m^2}\right]\right\}.$$

We now rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_R^2}{2}\phi^2 - \frac{\lambda\mu^\varepsilon}{3!}\phi^3 - \frac{1}{2}\delta m^2\phi^2 \tag{2.118}$$

with $\delta m^2 = m^2 - m_R^2$. This amounts to the replacement $m^2 \to m_R^2$ and add the term $\frac{1}{2}\delta m^2\phi^2$as a new vertex. The new self-energy $\Sigma^R(p)$ is then

$$\Sigma^R(p, m_R) = \Sigma(p, m_R) + \delta m^2. \tag{2.119}$$

Now if we choose

$$\delta m^2 = \frac{\lambda_0^2}{2}\frac{1}{(4\pi)^2}\left[-\frac{1}{\varepsilon} + c_m\right] \tag{2.120}$$

where $c_m$ is finite for $\varepsilon \to 0$, but is otherwise arbitrary (different renormalization schemes correspond to different choices of $c_m$), then the pole at $\varepsilon = 0$ cancels out and

$$\Sigma^R(p, m_R) = \frac{\lambda_0^2}{2}\frac{1}{(4\pi)^2}\left\{c_m - \gamma + \ln\left(\frac{4\pi\mu^2}{m_R^2}\right) - \int_0^1 d\alpha \ln\left[\frac{m_R^2-\alpha(1-\alpha)p^2-i\varepsilon}{m_R^2}\right]\right\}$$

is finite.

We now study this choice of $c_m$ in various renormalization schemes.

**(i) Momentum subtraction** Suppose we choose

$$\Sigma^R(p, m_R)|_{p^2=-M^2} = 0 \tag{2.121}$$

for some $M^2$. (This corresponds to normalizing the propagator such that it is the same as the free propagator at $p^2 = -M^2$.) Then the constant $c_m$ is given by

$$c_m = \gamma - \ln\left(\frac{4\pi\mu^2}{m_R^2}\right) + \int_0^1 d\alpha \ln\left[\frac{m_R^2 + \alpha(1-\alpha)M^2 - i\varepsilon}{m_R^2}\right] \tag{2.122}$$

and the self-energy is

$$\Sigma^R(p, m_R) = \frac{\lambda_0^2}{2}\frac{1}{(4\pi)^2}\int_0^1 d\alpha \ln\left[\frac{m_R^2 - \alpha(1-\alpha)p^2 - i\varepsilon}{m^2 + \alpha(1-\alpha)M^2 - i\varepsilon}\right]. \tag{2.123}$$

**(ii) Minimum subtraction (MS scheme)** This corresponds to the choice $c_m = 0$, and it means that we subtract only the pole at $\varepsilon = 0$ and the self-energy is

$$\Sigma^R(p, m_R) = \frac{\lambda_0^2}{2}\frac{1}{(4\pi)^2}\left\{-\gamma + \ln\left(\frac{4\pi\mu^2}{m_R^2}\right) - \int_0^1 d\alpha \ln\left[\frac{m_R^2 - \alpha(1-\alpha)p^2 - i\varepsilon}{m_R^2}\right]\right\}.$$

**(iii) Modified minimum substraction ($\overline{\mathbf{MS}}$ scheme)** It turns out that the combination

$$(1/\varepsilon) - \gamma + \ln 4\pi \tag{2.124}$$

always appears in the dimensional regularization. Thus it is convenient to choose

$$c_m = \gamma - \ln 4\pi \tag{2.125}$$

and we get

$$\Sigma^R(p, m_R) = -\frac{\lambda_0^2}{2}\frac{1}{(4\pi)^2}\int_0^1 d\alpha \ln\left[\frac{m_R^2 - \alpha(1-\alpha)p^2 - i\varepsilon}{\mu^2}\right]. \tag{2.126}$$

**(c-II) The tadpole diagram**

We have

$$i\tau = \frac{(i\lambda_0\mu^\varepsilon)}{2}\int \frac{d^n k}{(2\pi)^n}\frac{i}{k^2 - m_R^2} = \frac{i}{2}\frac{\lambda_0\mu^\varepsilon}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon - 1)(m_R)^{1-\varepsilon}. \tag{2.127}$$

This will contribute to the vacuum expectation value

$$\langle 0|\phi(0)|0\rangle = \frac{\tau}{m_R^2} \tag{2.128}$$

and will give an infinite constant vacuum expectation value for the field. To eliminate this infinity, we can add another counterterm to the Lagrangian of the form

$$\mathcal{L}_{tad} = -\tau\phi. \tag{2.129}$$

This counterterm will have the effect of cancelling all the tadpole terms, without interfering with any other consideration.

**Remark.** The parameter of renormalized mass, $m_R$, can be related to measurable quantities as follows. The physical mass of the $\phi$-particle, $m_p$, is defined to be the position of the single-particle pole in the two-point function. This gives $m_p$ as a function

$$m_p = m_p(m_R, \lambda, \mu, c_m). \tag{2.130}$$

We can solve for $m_R$ in terms of other parameters,

$$m_R = m_R(m_p, \lambda, \mu, c_m). \tag{2.131}$$

It is important to note that the parameter $m_R$ has implicit dependence on the arbitrary mass scale $\mu$.

## 2.8 Renormalization of composite operators

Consider a theory with a fermion and a complex scalar field. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2\phi^2 - \frac{\lambda}{2}(\phi^*\phi)^2 \\ + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + g\bar{\psi}\psi\phi + h.c.\end{aligned} \tag{2.132}$$

Show that the composite operators

$$O_1^\mu = \bar{\psi}\gamma^\mu\psi, \qquad O_2^\mu = i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) \tag{2.133}$$

mix under the renormalization.

### Solution to Problem 2.8

Vertices for the composite operators are displayed in Fig 2.6, where solid lines are fermions and dashed lines are bosons. Add two terms to the Lagrangian for these operators

$$\mathcal{L} \to \mathcal{L} + iJ_{1\mu}(x)O_1^\mu(x) + iJ_{2\mu}(x)O_2^\mu(x). \tag{2.134}$$

The one-loop divergent diagrams for these composite operators are shown in Fig. 2.7.

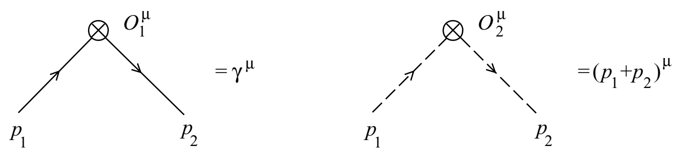



FIG. 2.6.

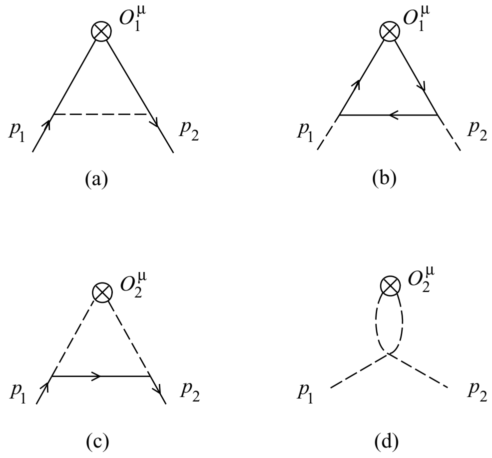



FIG. 2.7.

Figure 2.7(a) is a logarithmically divergent Green's function for the operator $O_1^\mu$ with two fermion external lines. We can separate out the divergent part as

$$\Gamma_{1a}^{\mu}(p_1, p_2) = \gamma^\mu \Gamma_{1a}(0,0) + p_{1\mu}\tilde{\Gamma} + \cdots \tag{2.135}$$

where the first term is logarithmically divergent and all other terms are finite. Thus we need a counterterm of the form

$$-i\Gamma_{1a}(0,0)\bar{\psi}\gamma^\mu\psi J_{1\mu}(x) = -i\Gamma_{1a}(0,0)J_{1\mu}O_1^\mu. \tag{2.136}$$

Figure 2.7( b) gives a logarithmically divergent Green's function for the operator $O_1^\mu$ with two scalar external lines. Again, separate out the divergent part as

$$\Gamma_{1b}^{\mu}(k_1, k_2) = (k_1 + k)^\mu \Gamma_{1b}(0,0) + \cdots \tag{2.137}$$

It shows the necessity of a counterterm of the form

$$-i\Gamma_{1b}(0,0)J_{1\mu}O_2^\mu. \tag{2.138}$$

Figure 2.7(c) shows the necessity of a counterterm of the form

$$-i\Gamma_{2c}(0,0)J_{2\mu}O_1^\mu. \tag{2.139}$$

Figure 2.7(d) shows the necessity of a counterterm of the form

$$-i\Gamma_{2d}(0,0)J_{2\mu}O_2^\mu. \tag{2.140}$$

Thus the effective Lagrangian which contains the composite operators and their one-loop counterterms is of the form

$$\begin{aligned}
\mathcal{L}_c &= iJ_{1\mu}O_1^\mu(1 - \Gamma_{1a}(0,0)) + iJ_{2\mu}O_2^\mu(1 - \Gamma_{2d}(0,0)) \\
&\quad - iJ_{1\mu}O_2^\mu\Gamma_{1b}(0,0) - iJ_{2\mu}O_1^\mu\Gamma_{1c}(0,0) \\
&= iJ_{1\mu}O_1^\mu Z_{11} + iJ_{2\mu}O_2^\mu Z_{22} + iJ_{1\mu}O_2^\mu Z_{12} + iJ_{2\mu}O_1^\mu Z_{21} \\
&= i\left(J_1^\mu,\ J_2^\mu\right)\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} O_1^\mu \\ O_2^\mu \end{pmatrix} \\
&= iJ_i^\mu Z_{ij} O_j^\mu
\end{aligned} \tag{2.141}$$

where

$$Z_{11} = 1 - \Gamma_{1a}(0,0), \qquad Z_{12} = -\Gamma_{1b}(0,0),$$
$$Z_{22} = 1 - \Gamma_{2d}(0,0), \qquad Z_{21} = -\Gamma_{2c}(0,0). \tag{2.142}$$

The renormalization constants are now in the matrix form, explicitly display the mixing of these operators,

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \tag{2.143}$$

which is neither symmetric nor real. Nevertheless we can diagonalize this by biunitary transformation (see CL-Section 11.3 for the details of biunitary transformation),

$$Z = U Z_d V^{\dagger} \tag{2.144}$$

where

$$Z_d = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \tag{2.145}$$

is diagonal. $V$ and $U$ are unitary matrices. Then we can write

$$\mathcal{L}_c = i J_i^{\mu} Z_{ij} O_j^{\mu} = i \bar{J}_1^{\mu} Z_1 \bar{O}_1^{\mu} + i \bar{J}_2^{\mu} Z_2 \bar{O}_2^{\mu} \tag{2.146}$$

where

$$\bar{O}_i^{\mu} = V_{ij}^{\dagger} O_j^{\mu}, \qquad \bar{J}_i^{\mu} = U_{ij}^{\dagger} J_j^{\mu}.$$

This means that neither $O_1^{\mu}$ nor $O_2^{\mu}$ are multiplicatively renormalizable. But the combinations

$$\bar{O}_1^{\mu} = V_{11}^{\dagger} O_1^{\mu} + V_{12}^{\dagger} O_2^{\mu}, \qquad \bar{O}_2^{\mu} = V_{21}^{\dagger} O_1^{\mu} + V_{22}^{\dagger} O_2^{\mu} \tag{2.147}$$

are multiplicatively renormalized.

## 2.9 Cutkosky rules

In the $\lambda\phi^3$ theory, the one-loop diagram in Fig. 2.5 gives the contribution

$$\Gamma(s) = \Gamma(p^2) = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(p+k)^2 - \mu^2 + i\varepsilon]} \frac{1}{[k^2 - \mu^2 + i\varepsilon]}. \tag{2.148}$$

Show that in the complex $s$ plane, the imaginary part for $s \geq 4\mu^2$ is of the form

$$\begin{aligned} Im\Gamma(s) &= \frac{1}{2}[\Gamma(s+i\varepsilon) - \Gamma(s-i\varepsilon)] \\ &= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} (-2\pi i)^2 \delta((p+k)^2 - \mu^2) \delta(k^2 - \mu^2). \end{aligned} \tag{2.149}$$

**Solution to Problem 2.9**

From the structure of the two propagators, the poles are located at the following two locations in the complex $k_0$ plane. From the second propagator, we have

$$k_0 = \pm\left(\mathbf{k}^2 + \mu^2\right)^{1/2} \mp i\varepsilon = \pm(E_k - i\varepsilon) \tag{2.150}$$

where $E_k = (\mathbf{k}^2 + \mu^2)^{1/2}$, and from the first propagator, we have

$$k_0 = -p_0 \pm ((\mathbf{p}+\mathbf{k})^2 + \mu^2)^{1/2} \mp i\varepsilon = -p_0 \pm (E_{p+k} - i\varepsilon). \tag{2.151}$$

The integrand is of the form

$$I \propto \frac{1}{(k_0 - E_k + i\varepsilon)} \frac{1}{(k_0 + E_k - i\varepsilon)} \frac{1}{(k_0 + p_0 + E_{p+k} - i\varepsilon)} \times \frac{1}{(k_0 + p_0 - E_{p+k} + i\varepsilon)}.$$

We can close the contour in the upper half plane and get the contribution from the residues at $k_0 = -E_k + i\varepsilon$ and $k_0 = -p_0 - E_{p+k} + i\varepsilon$.

**(i) Residue at** $k_0 = -E_k + i\varepsilon$

$$I_1 = -2\pi i \frac{1}{(2E_k - i\varepsilon)} \frac{1}{(-E_k + E_{p+k} + p_0 - i\varepsilon)} \frac{1}{(-E_k - E_{p+k} + p_0 + i\varepsilon)}. \tag{2.152}$$

The last two terms in the denominator can be put into the form

$$\begin{aligned}(E_k - p_0)^2 - E_{p+k}^2 &= \left[(\mathbf{k} + \mu^2)^{1/2} - p_0\right]^2 - (\mathbf{p}+\mathbf{k})^2 - \mu^2 \\ &= p^2 - 2p_0E_k - 2\mathbf{p}\cdot\mathbf{k} + i\varepsilon.\end{aligned}$$

For convenience we can take $p^\mu = (p_0, 0)$, then $p_0 = \sqrt{s}$ and

$$I_1 = \frac{1}{2E_k} \frac{1}{S - 2\sqrt{S}E_k + i\varepsilon}(-2\pi i). \tag{2.153}$$

Since $E_k \geq \mu$,we see that for $S \geq 4\mu^2$ the factor $(S - 2\sqrt{S}E_k + i\varepsilon)^{-1}$ has singularity along the path of integration (from $E_k = \mu$ to $\infty$) and will contribute to the discontinuity for $S \geq 4\mu^2$.

**(ii) Residue at** $k_0 = -p_0 - E_{p+k} + i\varepsilon$

$$I_2 = -2\pi i \frac{1}{(2E_{p+k} - i\varepsilon)} \frac{1}{(p_0 + E_{p+k} - E_k + i\varepsilon)} \frac{1}{(p_0 + E_{p+k} + E_k - i\varepsilon)}. \tag{2.154}$$

In the denominator, we have

$$\begin{aligned}(p_0 + E_{p+k})^2 - E_k^2 - i\varepsilon &= p_0^2 + (\mathbf{p}+\mathbf{k})^2 + \mu^2 + 2p_0E_{p+k} - \mathbf{k}^2 - \mu^2 \\ &= p_0^2 + 2p_0E_{p+k}.\end{aligned} \tag{2.155}$$

and

$$I_2 = -2\pi i \left(\frac{1}{2E_{p+k}}\right) \frac{1}{s + 2\sqrt{s}E_{p+k} + i\varepsilon}. \tag{2.156}$$

It is easy to see that this denominator never vanishes for $s \geq 4\mu^2$ and will not give discontinuity in the physical region. Thus if we close the integration contour in the upper half $k_0$ plane, only $I_1$ will contribute to the discontinuity in the physical region.

For the calculation of the discontinuity, we write

$$I_1 = (-2\pi i) \left(\frac{1}{2E_k}\right) \frac{1}{p(p - 2E_k + i\varepsilon)} \tag{2.157}$$

where $p = \sqrt{s}$. Using the formula

$$\frac{1}{x - a \pm i\varepsilon} = P\frac{1}{x - a} \mp i\pi\delta(x - a) \tag{2.158}$$

we can obtain the discontinuity across the cut for the case $s \geq 4\mu^2$,

$$\text{disc}\, I_1 = I_1(p + i\varepsilon) - I_1(p - i\varepsilon) = (-2\pi i)^2\delta(p - 2E_k)\left(\frac{1}{2E_k}\right)^2. \tag{2.159}$$

To get a more systematic rule for calculating the discontinuity, we write $I_1$ as

$$I_1 = \int dk_0\, (-2\pi i)\delta(k^2 - \mu^2)\frac{1}{(p + k)^2 - \mu^2 + i\varepsilon}. \tag{2.160}$$

We see that this corresponds to replacing the propagator $(k^2 - \mu^2 + i\varepsilon)^{-1}$ by $(-2\pi i)\delta(k^2 - \mu^2)$. Similarly,

$$\text{disc}\, I_1 = \int dk_0\, (-2\pi i)\delta(k^2 - \mu^2)(-2\pi i)\delta[(p + k)^2 - \mu^2] \tag{2.161}$$

or

$$\text{disc}\, \Gamma(s) = \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}(-2\pi i)\delta(k^2 - \mu^2)(-2\pi i)\delta[(p + k)^2 - \mu^2] \tag{2.162}$$

which is the requested result.

**Discussion**

As indicated by this calculation, we can obtain the discontinuity of $\Gamma(s)$ by putting each particle in the loop on the mass shell with the replacement of

$$\frac{1}{k^2 - \mu^2 + i\varepsilon} \longrightarrow (-2\pi i)\delta(k^2 - \mu^2)\theta(k_0). \tag{2.163}$$

In fact, this discontinuity can be written in terms of physical matrix elements, as we now illustrate:

$$\Gamma(s) = \frac{\lambda^2}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^4(k_1+k_2-p)\frac{1}{(k_1^2-\mu^2+i\varepsilon)}\frac{1}{(k_2^2-\mu^2+i\varepsilon)}.$$

Then using the replacement given above we get the discontinuity as

$$\begin{aligned}\operatorname{disc}\Gamma(s) &= \frac{\lambda^2 i}{2}\int \frac{d^4k_1}{(2\pi)^4}(-2\pi i)\delta(k_1^2-\mu^2)\theta(k_{10}) \\ &\times \int \frac{d^4k_2}{(2\pi)^4}(-2\pi i)\delta(k_2^2-\mu^2)(2\pi)^4 \\ &\times \delta^4(k_1+k_2-p)\theta(k_{20}). \end{aligned} \tag{2.164}$$

The integration over $k_0$ is of the form

$$\int \frac{dk_0}{2\pi}(-2\pi i)\delta(k^2-\mu^2)\theta(k_0) = \frac{-i}{2E_k} \tag{2.165}$$

where $E_k = (\mathbf{k}^2+\mu^2)^{1/2}$, and

$$\operatorname{disc}\Gamma(s) = \frac{1}{2}\int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}}\int \frac{d^3k_2}{(2\pi)^3 2E_{k_2}}(-i\lambda)^2(2\pi)^4\delta^4(k_1+k_2-p).$$

The factor $(-i\lambda)$ is just the scattering amplitude in first order of $\lambda$, $T_1 = -i\lambda$. Thus to order $\lambda^2$, the discontinuity of the scattering amplitude in the variable $s = p^2$ can be written as the integral over the phase space of $|T_1|^2$. This is the essence of the unitarity of the $S$-matrix, $SS^\dagger = S^\dagger S = 1$ which implies for the $T$-matrix $(S = 1 + iT)$

$$T - T^\dagger = TT^\dagger \tag{2.166}$$

or

$$T_{if} - T_{if}^* = \sum_n T_{in}T_{fn}^*. \tag{2.167}$$

The prescription of replacing $(k^2-\mu^2+i\varepsilon)^{-1}$ by $(-2\pi i)\delta(k^2-\mu^2)$ is a simple example of the *Cutkosky rule* which gives a general method for computing the discontinuity for an arbitrary Feynman diagram and is summarized below:

- Cut through the diagram in all possible ways such that the cut propagator can be put simultaneously on the mass shell for the kinematic region of interest (e.g. only for $s \geq 4\mu^2$ can both propagators be put on the mass shell).
- For each cut, replace the propagator $(k^2-\mu^2+i\varepsilon)^{-1}$ by $(-2\pi i)\delta(k^2-\mu^2)\theta(k_0)$ and perform the loop integration.
- Sum the contributions of all possible cuts.

# 3 Renormalization group

## 3.1 Homogeneous renormalization-group equation

Consider the $\lambda\phi^4$ theory in $d$-dimensional space-time, where the Lagrangian for $\varepsilon = 2 - (d/2)$ is given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\nu \phi\right)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 \\ &= \frac{1}{2}\left(\partial_\nu \phi_R\right)^2 - \frac{m_R^2}{2}\phi_R^2 - \frac{\lambda_R}{4!}\mu^\varepsilon \phi_R^4 + (\text{counterterms})\end{aligned} \tag{3.1}$$

where $\lambda_R$, $\phi_R$, and $m_R$ are renormalized quantities, and $\mu$ is the arbitrary mass scale one needs to introduce in the dimensional regularization.

Use the fact that the unrenormalized $n$-point Green's functions $\Gamma^{(n)}\left(p_i, \lambda_0, m_0\right)$ depend on the bare parameters $(m_0, \lambda_0)$ and are independent of the arbitrary mass scale, $\mu$, present in any scheme of dimensional regularization,

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)}\left(p_i, \lambda_0, m_0\right) = 0, \qquad \text{with } m_0, \lambda_0 \text{ held fixed} \tag{3.2}$$

to derive the renormalization group (RG) equation for this theory.

**Solution to Problem 3.1**

Recall that the relation, CL-eqn (3.50), between unrenormalized and renormalized Green's functions is given by

$$\Gamma^{(n)}\left(p_i, \lambda_0, m_0\right) = Z_\phi^{-n/2}\Gamma_R^{(n)}\left(p_i, \lambda_R, m_R, \mu\right) \tag{3.3}$$

Thus the statement of $\mu\partial/\partial\mu\Gamma^{(n)} = 0$ means that

$$\mu\frac{\partial}{\partial\mu}\left[Z_\phi^{-n/2}\Gamma_R^{(n)}\left(p_i, \lambda_R, m_R, \mu\right)\right] = 0. \tag{3.4}$$

Note that both $\lambda_R$ and $m_R$ depend implicitly on $\mu$. Thus we have

$$\begin{aligned}&\left[-\frac{n}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi + \mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda_R}{\partial\mu}\frac{\partial}{\partial\lambda_R} + \mu\frac{\partial m_R}{\partial\mu}\frac{\partial}{\partial m_R}\right] \\ &\quad\times\Gamma_R^{(n)}\left(p_i, \lambda_R, m_R, \mu\right) = 0.\end{aligned}$$

Defining the quantities,

$$\gamma\left(\lambda_R\right) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi, \qquad \beta\left(\lambda_R\right) = \mu\frac{\partial\lambda_R}{\partial\mu}, \qquad \gamma_m\left(\lambda_R\right)m_R = \mu\frac{\partial m_R}{\partial\mu}, \tag{3.5}$$

we can write the renormalization group equation as

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\left(\lambda_R\right)\frac{\partial}{\partial\lambda_R}-n\gamma\left(\lambda_R\right)+m_R\gamma_m\left(\lambda_R\right)\frac{\partial}{\partial m_R}\right]\Gamma_R^{(n)}\left(p_i,\lambda_R,m_R,\mu\right)=0. \tag{3.6}$$

**Remark 1.** Implicit in such calculation is the fact that the bare quantities $\lambda_0$ and $m_0$ are held fixed.

**Remark 2.** This equation is a homogeneous equation which is more convenient to work with than the original Callan–Symanzik equation.

## 3.2 Renormalization constants

In $\lambda\phi^4$ theory, the renormalized and unrenormalized coupling constants in the dimensional regularization scheme are related by

$$\lambda_R(\mu)=\mu^{-\varepsilon}\bar{Z}^{-1}(\mu)\lambda_0 \tag{3.7}$$

where $\bar{Z}$ is the coupling constant renormalization constant of the form

$$\bar{Z}^{-1}=Z_\lambda^{-1}Z_\phi^2 \tag{3.8}$$

where $Z_\lambda$ and $Z_\phi$ are defined in CL-eqns (2.23), (2.36), and (2.40).

**(a)** Show that the $\beta$-function can be written as

$$\beta\left(\lambda_R\right)=-\varepsilon\lambda_R-\frac{\mu}{\bar{Z}}\frac{d\bar{Z}}{d\mu}\lambda_R. \tag{3.9}$$

**(b)** In the one-loop approximation, we have

$$Z_\lambda^{-1}=1-\frac{3\lambda}{16\pi^2\varepsilon},\qquad Z_\phi=1+O\left(\lambda^2\right). \tag{3.10}$$

Show that

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}+O\left(\lambda^3\right). \tag{3.11}$$

**Solution to Problem 3.2**

**(a)** By explicit differentiation of eqn (3.7), we have

$$\begin{aligned}\beta\left(\lambda_R\right)&=\mu\frac{\partial\lambda_R}{\partial\mu}=\mu\frac{\partial}{\partial\mu}\left(\mu^{-\varepsilon}\bar{Z}(\mu)\lambda_0\right)\\&=-\varepsilon\mu^{-\varepsilon}\bar{Z}(\mu)\lambda_0-\mu^{-\varepsilon}\mu\frac{d\bar{Z}}{d\mu}\lambda_0\\&=-\varepsilon\lambda_R-\frac{\mu}{\bar{Z}}\frac{d\bar{Z}}{d\mu}\lambda_R.\end{aligned} \tag{3.12}$$

**(b)** Substituting the one-loop result (3.10) into eqn (3.8), we get

$$\bar{Z}^{-1} = Z_\lambda^{-1} Z_\phi^2 \simeq Z_\lambda^{-1} = 1 - \frac{3\lambda_R}{16\pi^2 \varepsilon} \tag{3.13}$$

or $\bar{Z} \simeq 1 + \left(3\lambda_R/16\pi^2\varepsilon\right)$. Thus

$$\mu \frac{d\bar{Z}}{d\mu} = \frac{3}{16\pi^2\varepsilon} \mu \frac{d\lambda_R}{d\mu} = \frac{3}{16\pi^2\varepsilon} \beta\left(\lambda_R\right) \tag{3.14}$$

and, ignoring higher-order terms, we obtain

$$\frac{1}{\bar{Z}} \mu \frac{d\bar{Z}}{d\mu} \simeq \frac{3}{16\pi^2\varepsilon} \beta\left(\lambda_R\right). \tag{3.15}$$

In this way the relation (3.9) becomes

$$\beta\left(\lambda_R\right) = -\varepsilon\lambda_R - \left(\frac{\mu}{\bar{Z}} \frac{d\bar{Z}}{d\mu}\right) \lambda_R = -\varepsilon\lambda_R - \frac{3\lambda_R}{16\pi^2\varepsilon} \beta\left(\lambda_R\right). \tag{3.16}$$

Solving for $\beta\left(\lambda_R\right)$, we get

$$\beta\left(\lambda_R\right) = -\varepsilon\lambda_R \left(1 + \frac{3\lambda_R}{16\pi^2\varepsilon}\right)^{-1} \simeq -\varepsilon\lambda_R \left(1 - \frac{3\lambda_R}{16\pi^2\varepsilon}\right). \tag{3.17}$$

Taking the limit $\varepsilon \to 0$, we obtain the stated result:

$$\beta\left(\lambda_R\right) = \frac{3\lambda_R^2}{16\pi^2} + O\left(\lambda_R^3\right). \tag{3.18}$$

**Remark.** More general analysis of $\varepsilon$ dependence of $\beta(\lambda)$ can be carried out as follows. We first write eqn (3.7) as

$$\lambda_0 = \lambda(\mu)\mu^\varepsilon \bar{Z}. \tag{3.19}$$

From $\mu d\lambda_0/d\mu = 0$, we get

$$\mu^\varepsilon \left[\varepsilon(\lambda\bar{Z}) + \mu\frac{d}{d\mu}(\lambda\bar{Z})\right] = 0 \qquad \text{or} \qquad -\varepsilon(\lambda\bar{Z}) = \mu\frac{d}{d\mu}(\lambda\bar{Z}). \tag{3.20}$$

In the $\overline{\text{MS}}$ scheme , $\bar{Z}$ can be written as power series in $(1/\varepsilon)$,

$$\bar{Z} = 1 + \frac{b_1(\lambda)}{\varepsilon} + \frac{b_2(\lambda)}{\varepsilon^2} + \cdots \tag{3.21}$$

then

$$\lambda\bar{Z} = \lambda + \frac{a_1(\lambda)}{\varepsilon} + \frac{a_2(\lambda)}{\varepsilon^2} + \cdots \tag{3.22}$$

where $a_i(\lambda) = \lambda b_i(\lambda)$. Differentiating both sides of this equation, we get

$$\mu \frac{d}{d\mu}(\lambda \bar{Z}) = \left[1 + \sum_{n=1} \frac{da_n}{d\lambda} \frac{1}{\varepsilon^n}\right] \mu \frac{d\lambda}{d\mu} = \left[1 + \sum_{n=1} \frac{da_n}{d\lambda} \frac{1}{\varepsilon^n}\right] \beta(\lambda). \tag{3.23}$$

Equation (3.20) becomes

$$-\varepsilon \left[\lambda + \frac{a_1(\lambda)}{\varepsilon} + \frac{a_2(\lambda)}{\varepsilon^2} + \cdots\right] = \beta(\lambda) \left[1 + \frac{da_1}{d\lambda} \frac{1}{\varepsilon} + \frac{da_2}{d\lambda} \frac{1}{\varepsilon^2} + \cdots\right]. \tag{3.24}$$

Assume that the $\beta(\lambda)$ is a finite series in $\varepsilon$,

$$\beta(\lambda) = \left[\beta_0 + \beta_1 \varepsilon + \beta_2 \varepsilon^2 + \cdots + \beta_M \varepsilon^M\right]. \tag{3.25}$$

By identifying powers of $\varepsilon$ on both sides eqn (3.24), we see that the $\beta(\lambda)$ series must terminate after the first power of $\varepsilon$ ($\beta_k = 0$ for $k > 1$):

$$\beta(\lambda) = \beta_0 + \beta_1 \varepsilon. \tag{3.26}$$

From eqn (3.24) we then get

$$\beta_1 = -\lambda \tag{3.27}$$

and

$$\beta_0 + \beta_1 \frac{da_1}{d\lambda} = -a_1 \qquad \text{or} \qquad \beta_0 = -a_1 + \lambda \frac{da_1}{d\lambda}. \tag{3.28}$$

Thus the $\beta$-function is given by

$$\beta(\lambda) = -\lambda\varepsilon - a_1 + \lambda \frac{da_1}{d\lambda} \to -a_1 + \lambda \frac{da_1}{d\lambda} \qquad \text{as} \qquad \varepsilon \to 0. \tag{3.29}$$

Using $a_1 = \lambda b_1$, we get

$$\beta(\lambda) = \lambda^2 \left(\frac{db_1}{d\lambda}\right). \tag{3.30}$$

In fact, eqn (3.24) also relates different powers of $(1/\varepsilon)$ in the expansion of $\bar{Z}$(or $\lambda\bar{Z}$). From the coefficient of $(1/\varepsilon)^n$ we have

$$-a_{n+1} = \beta_0 \frac{da_n}{d\lambda} + \beta_1 \frac{da_{n+1}}{d\lambda} = \left(-a_1 + \lambda \frac{da_1}{d\lambda}\right) \frac{da_n}{d\lambda} - \lambda \frac{da_{n+1}}{d\lambda}$$

or

$$-a_{n+1} + \lambda \frac{da_{n+1}}{d\lambda} = \left(-a_1 + \lambda \frac{da_1}{d\lambda}\right) \frac{da_n}{d\lambda}. \tag{3.31}$$

Thus all the coefficients $a_n$ with $n > 1$ can be determined from $a_1$, by repeated use of eqn (3.31).

## 3.3 $\beta$-function for QED

The photon propagator is given by

$$ie^2 D_{\mu\nu}(q) = \frac{-i}{q^2}\frac{e_0^2}{1+\pi(q)}g_{\mu\nu} \tag{3.32}$$

where

$$e^2 = \frac{e_0^2}{1+\pi(0)} = Z_3 e_0^2, \qquad Z_3 = \frac{1}{1+\pi(0)} \simeq 1 - \pi(0), \tag{3.33}$$

and

$$\pi(q)\left(g_{\mu\nu} - q_\mu q_\nu\right) = \pi_{\mu\nu}(q) \tag{3.34}$$

is the vacuum polarization tensor. Define the running electric charge as

$$e^2(q) \equiv \frac{e^2}{1 + \operatorname{Re}\tilde{\pi}\left(q^2\right)} \tag{3.35}$$

where $\tilde{\pi}(q^2) = \pi(q^2) - \pi(0)$ is finite. Show that in one-loop, we have

$$e^2(-\mu_R'^2) = e^2(-\mu_R^2) + \frac{e_0^4}{12\pi^2}\ln\frac{\mu_R'^2}{\mu_R^2} \tag{3.36}$$

for the case of $\mu_R^2,\ \mu_R'^2 \gg m^2$. If we define

$$\beta(e) = \mu_R \frac{\partial e}{\partial \mu_R}, \tag{3.37}$$

then we will have

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5). \tag{3.38}$$

**Solution to Problem 3.3**

Vacuum polarization in QED is calculated in Problem 2.6, and is given by

$$\begin{aligned}\pi(q^2) &= \frac{e_0^2}{2\pi^2}\frac{\Gamma(\varepsilon/2)}{(4\pi)^{-\varepsilon/2}}\mu^\varepsilon \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{[m^2 - q^2\alpha(1-\alpha)]^{\varepsilon/2}} \\ &= \frac{e_0^2}{12\pi^2}\left\{\left(\frac{1}{\varepsilon} + \ln 4\pi - \gamma\right)\right. \\ &\quad \left. - 6\int_0^1 d\alpha\, \alpha(1-\alpha)\ln\left[\frac{m^2 - q^2\alpha(1-\alpha)}{\mu^2}\right] + O(\varepsilon)\right\}. \end{aligned} \tag{3.39}$$

Taking $\mu = m$, the subtracted quantity becomes

$$\begin{aligned}\tilde{\pi}(q^2) &= \pi(q^2) - \pi(0) \\ &= -\frac{e_0^2}{2\pi^2}\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2 - q^2\alpha(1-\alpha)}{m^2}\right]. \end{aligned} \tag{3.40}$$

Write

$$e^2(q^2) = \frac{e^2}{1 + \mathrm{Re}\,\tilde{\pi}(q^2)} \simeq e^2\left[1 - \mathrm{Re}\,\tilde{\pi}(q^2)\right]. \tag{3.41}$$

For the case of $|q^2| \gg m^2$, we have

$$\tilde{\pi}(q^2) \simeq -\frac{e_0^2}{12\pi^2}\ln\left(\frac{-q^2}{m^2}\right) \tag{3.42}$$

$$e^2(q^2) \simeq e^2\left[1 + \frac{e_0^2}{12\pi^2}\ln\left(\frac{-q^2}{m^2}\right)\right]. \tag{3.43}$$

Thus, as $(-q^2)$ increases, $e^2(q^2)$ also increases.

### Different subtraction schemes

From the vacuum polarization $\pi(q^2)$ given in eqn (3.39) we can illustrate the difference of different renormalization schemes.

**Momentum subtraction scheme.** In this scheme, we make a subtraction at $q^2 = -M^2$, then

$$\pi_R\left(q^2, M\right) = \frac{-e_0^2}{2\pi^2}\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2 - q^2\alpha(1-\alpha)}{m^2 + M^2\alpha(1-\alpha)}\right]. \tag{3.44}$$

Suppose $m^2 \gg \left|q^2\right|$ and $M^2$, then

$$\pi_R\left(q^2, M\right) \to \frac{e_0^2}{12\pi^2} O\left(\frac{q^2}{m^2}, \frac{M^2}{m^2}\right) \to 0. \tag{3.45}$$

This means that a heavy fermion will decouple in the vacuum polarization at energies much smaller than the heavy fermion mass. This property will enable us to ignore all the unknown particles which are much heavier than the present energies.

**$\overline{\text{MS}}$ scheme.** Here we subtract out the pole at $\varepsilon = 0$ and some constants,

$$\pi_R^{\text{MS}}(q^2) = \frac{-e_0^2}{2\pi^2}\int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left[\frac{m^2 - q^2\alpha(1-\alpha)}{\mu^2}\right]. \tag{3.46}$$

In the limit $m^2 \gg |q^2|$ we get

$$\pi_R^{\text{MS}}(q^2) \to \frac{-e_0^2}{12\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) \tag{3.47}$$

which is non-zero. Thus in this scheme, the heavy particles do not decouple at low energies. One way to remove the effect of the heavy particles is to integrate out the heavy fields in the Lagrangian and work with the effective Lagrangian without the heavy particles.

## 3.4 Behaviour of $\bar{g}$ near a simple fixed point

Derive the ultraviolet behaviour of $\bar{g}(t)$ in the case that the $\beta$-function is given by

$$\beta(g) = g(a^2 - g^2) \tag{3.48}$$

with $a$ being a known constant. This example illustrates the typical behaviour of the running coupling near a simple fixed point.

**Solution to Problem 3.4**

To analyse the asymptotic behaviour, we plot $\beta(g)$ vs. $g$,

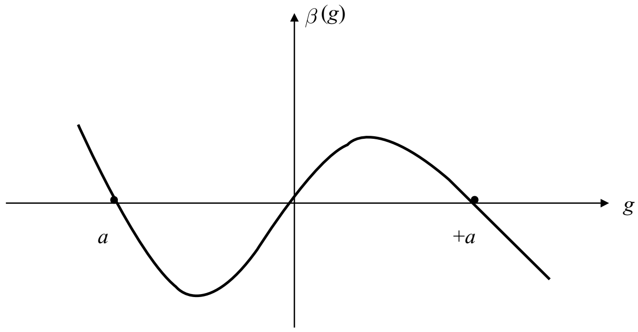



FIG. 3.1.

The initial condition for the running coupling constant is

$$\bar{g}(t) \to g_0 \qquad \text{at} \quad t = 0.$$

Then it is clear from this plot that

$$\begin{aligned} \bar{g}(t) &\to a && \text{if} \quad g_0 > 0 \\ \bar{g}(t) &\to -a && \text{if} \quad g_0 < 0. \end{aligned}$$

This can be verified by more explicit calculation as given below.

$$\frac{d\bar{g}}{dt} = \bar{g}\left(a^2 - \bar{g}^2\right) \;\Rightarrow\; \int \frac{d\bar{g}}{\left(a^2 - \bar{g}^2\right)} \frac{1}{\bar{g}} = \int dt. \tag{3.49}$$

Carrying out the integration and using the initial condition, we get

$$\frac{1}{2a^2}\left[\ln\left(\frac{\bar{g}^2}{\bar{g}^2 - a^2}\right) - \ln\left(\frac{g_0^2}{g_0^2 - a^2}\right)\right] = t \tag{3.50}$$

or

$$\frac{g_0^2\left(\bar{g}^2-a^2\right)}{\bar{g}^2\left(g_0^2-a^2\right)}=e^{-2a^2t} \tag{3.51}$$

so that

$$\bar{g}^2=\frac{a^2}{1-Ae^{-2a^2t}} \qquad \text{with} \qquad A=\frac{g_0^2-a^2}{g_0^2}. \tag{3.52}$$

Taking the square root, we get

$$\bar{g}=\frac{\pm a}{\left(1-Ae^{-2a^2t}\right)^{1/2}}. \tag{3.53}$$

To choose the sign, we need to go back to the initial condition that $\bar{g}=g_0$ at $t=0$. For the case $g_0>0$, we take the positive sign

$$\bar{g}=\frac{a}{\left(1-Ae^{-2a^2t}\right)^{1/2}}$$

so that at $t=0$,

$$\bar{g}=\frac{a}{(1-A)^{1/2}}=\frac{a}{(a/g_0)}=g_0.$$

In this case, $\bar{g}\to a$, as $t\to\infty$. For the case $g_0<0$, we need to choose the other sign

$$\bar{g}=\frac{-a}{\left(1-Ae^{-2a^2t}\right)^{1/2}}. \tag{3.54}$$

Then we have $\bar{g}\to -a$ as $t\to\infty$.

## 3.5 Running coupling near a general fixed point

At the stable critical point $g=g_0$, show that

**(a)** if $\beta(g)$ has a simple zero: $\beta(g)=-b(g-a)$ with $b>0$, then the approach of $g(t)$ to $g_0$ as $t\to\infty$ is exponential in $t$;

**(b)** if $\beta(g)$ has a double or higher zero: $\beta(g)=-b(g-a)^n$ with $b>0$ and $n>1$, then the approach of $g(t)$ to $g_0$ as $t\to\infty$ is some inverse power in $t$.

### Solution to Problem 3.5

**(a) Simple zero:** $\beta(g)=-b(g-a)$
From the renormalization group equation

$$\frac{d\bar{g}}{dt}=\beta(\bar{g}), \qquad \int\frac{d\bar{g}}{\bar{g}-a}=-\int b\,dt, \tag{3.55}$$

then with the initial condition of $\bar{g} \to g_0$ at $t = 0$, we have

$$\ln\left(\frac{\bar{g} - a}{g_0 - a}\right) = -bt \tag{3.56}$$

with $b > 0$,

$$\bar{g} = a + (g_0 - a)\, e^{-bt}. \tag{3.57}$$

Then, $\bar{g} \to a$ exponentially in the asymptotic $t \to \infty$ limit.

**(b) Double zero or higher:** $\beta(g) = -b(g - a)^n,\ n > 1$.
The same calculation yields

$$\int \frac{d\bar{g}}{(\bar{g} - a)^n} = -\int b dt \quad \Rightarrow \quad \frac{1}{(n-1)}\left[\frac{1}{(\bar{g} - a)^{n-1}} - \frac{1}{(g_0 - a)^{n-1}}\right] = bt$$

or

$$\bar{g} = a + \left[\frac{1}{1/(g_0 - a)^{n-1} + (n-1)bt}\right] \underset{t\to\infty}{\to} a + O\left(t^{-1/(n-1)}\right). \tag{3.58}$$

## 3.6 One-loop renormalization-group equation in massless $\lambda\phi^4$ theory

In the renormalization of the massless $\lambda\phi^4$ theory, we can momentum subtract at $p^2 = -M^2$ to avoid infrared singularities. In this case the renormalization-group equation takes the form

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} - n\gamma(\lambda)\right]\Gamma_R^{(n)}(p_1, p_2, \ldots, p_n) = 0. \tag{3.59}$$

Verify explicitly, the one-loop result for the four-point function $\Gamma_R^{(4)}(p_1, p_2, p_3)$ satisfies this renormalization-group equation.

**Solution to Problem 3.6**

From CL-eqn.(2.31), the four-point function in one-loop is of the form

$$\Gamma_0^{(4)}(s, t, u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u) \tag{3.60}$$

where $\Gamma(p^2)$ in the dimensional regularization scheme is given by [Cf. CL-eqn (2.121)]

$$\Gamma(p^2) = \frac{i\lambda_0^2}{32\pi^2}\left\{\frac{2}{4-d} - \int_0^1 d\alpha[\ln\alpha(1-\alpha)] - \ln(-p^2)\right\}. \tag{3.61}$$

Suppose we make a subtraction at some space-like momentum $p^2 = -M^2$. Then we have

$$\tilde{\Gamma}(p^2) = \Gamma(p^2) - \Gamma(-M^2) = \frac{-i\lambda_0^2}{32\pi^2}\ln\left(\frac{-p^2}{M^2}\right) \tag{3.62}$$

where the dependence on $M$ is rather simple (compared with the $\mu^2$ dependence in the massive theory). The renormalized Green's function being

$$\Gamma_R^{(4)}(s,t,u) = -i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \tag{3.63}$$

we get

$$M\frac{\partial}{\partial M}\Gamma_R^{(4)}(s,t,u) = \frac{3i\lambda_0^2}{16\pi^2} = \frac{3i\lambda^2}{16\pi^2} + O(\lambda^3). \tag{3.64}$$

Also we have

$$\beta(\lambda)\frac{\partial}{\partial\lambda}\Gamma_R^{(4)}(s,t,u) = \left(\frac{3\lambda^2}{16\pi^2}\right)[-i + O(\lambda)] \tag{3.65}$$

where we have used CL-eqn (3.47)

$$\beta(\lambda) = \left(\frac{3\lambda^2}{16\pi^2}\right) + O(\lambda^3). \tag{3.66}$$

Therefore, from CL-eqn (3.48) that $\gamma(\lambda) \simeq O(\lambda^2)$, we see that $\Gamma_R^{(4)}(s,t,u)$ satisfies the renormalization-group equation to order $\lambda^2$.

## 3.7 $\beta$-function for the Yukawa coupling

The Lagrangian for the Yukawa interaction is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + f\bar{\psi}\psi\phi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2. \tag{3.67}$$

Compute the Callan–Symanzik $\beta$-function for the coupling constant $f$.

**Solution to Problem 3.7**

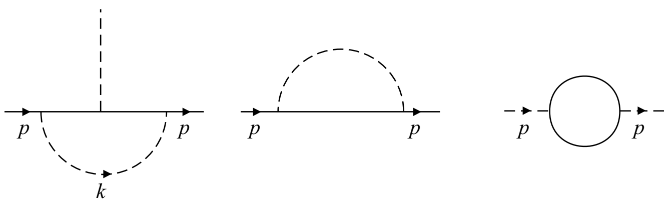



FIG. 3.2.

It is convenient to set all the masses to zero.

**(i) Vertex correction**

$$\Gamma = (-if)^3 \int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{\not{p} - \not{k}}\right)^2 \frac{i}{k^2}$$
$$= f^3 \int \frac{d^4k}{(2\pi)^4}\int \frac{d\alpha}{\left(k^2 - a^2\right)^2} \qquad \text{with } a^2 = -\alpha(1-\alpha)p^2 \tag{3.68}$$

where we have combined the denominators by using the Feynman parameter and have shifted the integration variable $k \to k + \alpha p$. The divergent part is then

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - a^2)^2} = \frac{i}{16\pi^2} \left( \ln \frac{\Lambda^2}{a^2} + \cdots \right) \tag{3.69}$$

so that

$$\Gamma = \frac{if^3}{16\pi^2} \left( \ln \frac{\Lambda^2}{a^2} + \cdots \right). \tag{3.70}$$

The vertex renormalization constant is then

$$Z_f = 1 + \frac{f^2}{16\pi^2} (\ln \Lambda^2 + \cdots). \tag{3.71}$$

Recall that the $\beta$-function is given by

$$\beta_f = -f \frac{\partial}{\partial(\ln \Lambda)} \bar{Z}_f, \qquad \text{where} \quad \bar{Z}_f = Z_f^{-1} Z_\phi Z_\psi^2. \tag{3.72}$$

Here $Z_\phi$ and $Z_\psi$ are the wavefunction renormalization constants for scalar and fermion fields. Thus the contribution coming from $Z_f$ is

$$\beta_1 = -f \frac{\partial}{\partial(\ln \Lambda)} Z_f^{-1} = \frac{f^3}{16\pi^2}(2). \tag{3.73}$$

**(ii) Fermion self-energy**

$$\Sigma_\psi(p) = (-if)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i(\not{p} - \not{k})}{(p-k)^2}. \tag{3.74}$$

Combine the denominators in the usual way

$$\frac{1}{k^2} \frac{1}{(p-k)^2} = \int \frac{d\alpha}{A^2} \tag{3.75}$$

with $A = (k - \alpha p)^2 - a^2$ and $a^2 = -\alpha(1-\alpha)p^2$. Shift the integration variable $k \to k + \alpha p$; the numerator becomes $(1-\alpha)\not{p} - \not{k}$. Then we have

$$\begin{aligned} \Sigma_\psi(p) &= f^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \frac{[(1-\alpha)\not{p} - \not{k}]}{(k^2 - a^2)} \\ &\to f^2 \int_0^1 d\alpha\, (1-\alpha)\not{p} \frac{i}{16\pi^2} \left( \ln \frac{\Lambda^2}{a^2} + \cdots \right) \end{aligned} \tag{3.76}$$

and the wave function renormalization constant is

$$Z_\psi = 1 - \frac{f^2}{32\pi^2} \ln \Lambda^2 + \cdots \tag{3.77}$$

and its contribution to the $\beta$-function is

$$\beta_2 = -f \frac{\partial}{\partial(\ln \Lambda)} Z_\psi = \frac{f^3}{16\pi^2}. \tag{3.78}$$

Since there are two such diagrams in the vertex, this contribution should be multiplied by two.

**(iii) Scalar self-energy**

$$\Sigma_\phi(p) = (-if)^2\,(-)\int \frac{d^4k}{(2\pi)^4} Tr\left(\frac{i}{\not k}\frac{i}{(\not p - \not k)}\right)$$
$$= (-)f^2 \int \frac{d^4k}{(2\pi)^4}\frac{Tr[\not k(\not p - \not k)]}{k^2(p-k)^2}. \tag{3.79}$$

The numerator is, in the dimensional regularization,

$$N = Tr[\not k(\not p - \not k)] = d\left(p\cdot k - k^2\right). \tag{3.80}$$

The denominator is

$$\frac{1}{k^2(p-k)^2} = \int_0^1 d\alpha \frac{1}{\left(k^2 - a^2\right)^2} \qquad \text{with} \quad a^2 = -\alpha(1-\alpha)p^2 \tag{3.81}$$

where the shift $k \to k + \alpha p$ has been made. With the shift the numerator is then

$$N = d[(k+\alpha p)^2 - p\cdot(k+\alpha p)] \to d[k^2 - \alpha(1-\alpha)p^2] = d(k^2 + a^2). \tag{3.82}$$

The self-energy is

$$\Sigma_\phi(p) = -f^2 \int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha \frac{d}{(k^2-a^2)^2}(k^2+a^2). \tag{3.83}$$

Using the formulae

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-a^2)^n} = \frac{i(-)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{a^2}\right)^{n-d/2} \tag{3.84}$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-a^2)^n} = \frac{i(-)^{n-1}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\left(\frac{1}{a^2}\right)^{n-d/2-1}\left(\frac{d}{2}\right) \tag{3.85}$$

we get

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k^2+a^2)}{(k^2-a^2)^2} = \frac{i}{(4\pi)^{d/2}}\left(\frac{1}{a^2}\right)^{1-d/2}\left[-\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right) + \Gamma\left(2-\frac{d}{2}\right)\right]$$
$$= \frac{i}{(4\pi)^{d/2}}\left(\frac{1}{a^2}\right)^{1-d/2}\left(\frac{1-d}{1-d/2}\right)\Gamma\left(2-\frac{d}{2}\right). \tag{3.86}$$

The self-energy is then

$$\Sigma_\phi(p) = -f^2\int_0^1 d\alpha \frac{di}{(4\pi)^{d/2}}\left(\frac{1-d}{1-d/2}\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{a^2}\right)^{1-d/2}. \tag{3.87}$$

The divergent part, which is relevant to the wavefunction renormalization, is then, with $d \to 4$ and $\Gamma(2 - d/2) \to (2/(4 - d))$,

$$\Sigma_\phi(p)_{div} = \frac{-ip^2 f^2}{16\pi^2} 2 \left( \frac{2}{4-d} \right) \to \frac{-ip^2 f^2}{16\pi^2} 2 \ln \frac{\Lambda^2}{\mu^2} \tag{3.88}$$

where we have used the correspondence $(2/(4-d)) \to \ln \Lambda^2$ (see CL-p. 56). The wave function renormalization constant is

$$Z_\phi = 1 + \frac{f^2}{16\pi^2} 2 \ln \frac{\Lambda^2}{\mu^2} \tag{3.89}$$

and its contribution to the $\beta$-function is

$$\beta_3 = \frac{f^3}{16\pi^2} 2. \tag{3.90}$$

The total contribution to the $\beta$-function is then

$$\beta = (\beta_1 + 2\beta_2 + \beta_3) = \frac{f^3}{16\pi^2} \left( 2 + 2 \times \frac{1}{2} + 2 \right) = \frac{5f^3}{16\pi^2}. \tag{3.91}$$

This is the result given in CL-eqn (10.16).

## 3.8 Solving the renormalization-group equation by Coleman's method

Consider a one-dimensional fluid with velocity $v(x)$ and in the fluid there are bacteria (see Coleman 1985). Let $\rho(t, x)$ and $g(x)$ be the density and the growth rate of the bacteria, respectively.

**(a)** Show that the density of the bacteria $\rho(t, x)$ satisfies the differential equation

$$\left[ \frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} - g(x) \right] \rho(t, x) = 0. \tag{3.92}$$

**(b)** The position of a fluid element is described by $\bar{x} = \bar{x}(t, x)$ with the initial condition $\bar{x}(0, x) = x$. Namely, the fluid element which was at $x$ at $t = 0$ is now at $\bar{x}$ at time $t$. Clearly $\bar{x}(t, x)$ satisfies the differential equation

$$\frac{d}{dt} \bar{x}(t, x) = v(x). \tag{3.93}$$

Show that if $\rho(0, x) = \rho_0(x)$, then at later time $\rho(t, x)$ is given by

$$\rho(t, x) = \rho_0(\bar{x}(t, x)) \exp \left[ \int_0^t dt' g(\bar{x}(t', x)) \right]. \tag{3.94}$$

**Solution to Problem 3.8**

**(a)** The term due to the growth rate $g(x)$ is self-evident. We will concentrate on the second term which is due to the motion of the fluid.

Consider a fluid element $f$ with length $dx$ located at $x$. The bacteria in this fluid element is just $\rho(t, x)\, dx$. At a later time, $t + \Delta t$, this fluid element is replaced by those which were located at $(x - v\Delta t)$ at time $t$. Thus the rate of change in the bacteria density in $f$ is

$$\frac{[\rho(t, x) - \rho(t, x - v\Delta t)]}{\Delta t} \simeq v\frac{\partial \rho}{\partial x} \tag{3.95}$$

where we have made the approximation $\rho(t, x - v\Delta t) \simeq \rho(t, x) - v\partial\rho/\partial x \Delta t$. This gives the second term in the differential equation.

**(b)** Integrating eqn (3.93) for $\bar{x}$, we get

$$\int_x^{\bar{x}} \frac{dy}{v(y)} = \int_0^t dt'. \tag{3.96}$$

We can differentiate this equation with respect to $x$ to get

$$\frac{1}{v(\bar{x})}\frac{\partial \bar{x}}{\partial x} - \frac{1}{v(x)} = 0. \tag{3.97}$$

Then for any function $f(x)$ we can show that

$$\begin{aligned} \frac{\partial}{\partial t} f(\bar{x}(t, x)) &= f'(\bar{x})\frac{d\bar{x}}{dt} = f'(\bar{x})v(\bar{x}), \\ v(x)\frac{\partial}{\partial x} f(\bar{x}(t, x)) &= v(x)f'(\bar{x})\frac{\partial \bar{x}}{\partial x} = f'(\bar{x})v(\bar{x}). \end{aligned} \tag{3.98}$$

Combining these we get

$$\left[\frac{\partial}{\partial t} - v(x)\frac{\partial}{\partial x}\right] f(\bar{x}(t, x)) = 0. \tag{3.99}$$

Or changing $t \to -t$, we get

$$\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right] f(\bar{x}(-t, x)) = 0. \tag{3.100}$$

Then it is straightforward to verify that the solution is of the form

$$\rho(t, x) = \rho_0(\bar{x}(-t, x)) \exp\left[\int_0^t dt' g(\bar{x}(-t', x))\right]. \tag{3.101}$$

**Remark.** The generalization to a higher dimension is simply

$$\left[\frac{\partial}{\partial t} + v_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i} - g(x_1, \ldots, x_n)\right] \rho(t, x_1, \ldots, x_n) = 0. \tag{3.102}$$

Define

$$\frac{d}{dt}\bar{x}_i(t, x_1, \ldots, x_n) = v_i(\bar{x}_1, \ldots, \bar{x}_n) \qquad \text{with} \quad \bar{x}_i(0, x_1, \ldots, x_n) = x_i. \tag{3.103}$$

The solution is then

$$\rho(t, x_1, \ldots, x_n) = \rho_0(\bar{x}_i(t, x_1, \ldots, x_n)) \exp\left[\int_0^t dt' g(\bar{x}_i(t', x_1, \ldots, x_n))\right]. \tag{3.104}$$

## 3.9 Anomalous dimensions for composite operators

In the $\lambda\phi^4$ theory, compute the anomalous dimensions for the composite operators, $\phi^2$ and $\phi^6$, in the one-loop approximation.

**Solution to Problem 3.9**

**(i) Anomalous dimension of $\phi^2$** As was described in CL-Section 2.4, the only one-loop divergent graph involving $\phi^2$ is in the two-point function $\Gamma^{(2)}_{\phi^2}$, and is of the form

$$\Gamma^{(2)}_{\phi^2}(p;\, p_1,\, p_2) = \left(\frac{-i\lambda}{2}\right)\int \frac{d^4 l}{(2\pi)^4}\left[\frac{i}{l^2-\mu^2}\right]\left[\frac{i}{(l-p)^2-\mu^2}\right]. \tag{3.105}$$

This has exactly the same structure as the function $\Gamma(p^2)$, given in CL-eqn (2.70), which appears in the four-point function. Taking over the result, we have

$$\Gamma^{(2)}_{\phi^2}(p;\, p_1,\, p_2) = \frac{-\lambda}{32\pi^2}\left\{\ln\frac{\Lambda^2}{\mu^2} - \int_0^1 d\alpha\, \ln[\mu^2 - \alpha(1-\alpha)p^2] + \cdots\right\} \tag{3.106}$$

and

$$Z_{\phi^2} = 1 + \Gamma^{(2)}_{\phi^2}(0,0,0) \simeq 1 - \frac{\lambda}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}. \tag{3.107}$$

The anomalous dimension is then

$$\gamma_{\phi^2} = -\frac{\partial}{\partial \ln\Lambda}\ln Z_{\phi^2} = \frac{\lambda}{16\pi^2}. \tag{3.108}$$

**(ii) Anomalous dimension of $\phi^6$** The one-loop divergent graphs are all of the type shown in Fig. 3.3 with altogether $\binom{6}{2}$ distinctive diagrams.

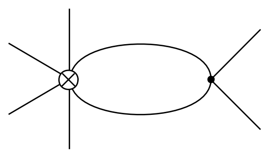


FIG. 3.3.

This again can be expressed in terms of $\Gamma(p^2)$. Taking into account the combinatorics we get

$$Z_{\phi^6} = 1 + 15\Gamma^{(2)}_{\phi^2}(0,0,0) \simeq 1 - \frac{15\lambda}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}. \tag{3.109}$$

The anomalous dimension is then

$$\gamma_{\phi^6} = \frac{15\lambda}{16\pi^2}. \tag{3.110}$$

# 4 Group theory and the quark model

## 4.1 Unitary and hermitian matrices

Show the following relationships between the unitary and hermitian matrices:

**(a)** Any $n \times n$ unitary matrix $U^\dagger U = 1$ can be written as

$$U = \exp(iH) \tag{4.1}$$

where $H$ is hermitian, $H^\dagger = H$.

**(b)** $\det U = 1$ implies that $H$ is traceless.

**Remark.** This result means that $n \times n$ unitary matrices with unit determinant can be generated by $n \times n$ traceless hermitian matrices.

### Solution to Problem 4.1

**(a)** The matrix $U$ can always be diagonalized by some unitary matrix $V$

$$VUV^\dagger = U_d \tag{4.2}$$

where $U_d$ is a diagonal matrix satisfying the unitarity condition $U_d U_d^\dagger = 1$. This implies that the each of the diagonal elements can be expressed as a complex number with unit magnitude $e^{i\alpha}$.

$$U_d = \begin{pmatrix} e^{i\alpha_1} & & & \\ & e^{i\alpha_2} & & \\ & & \ddots & \\ & & & e^{i\alpha_n} \end{pmatrix} \tag{4.3}$$

where $\alpha_i$s are real. It is then straightforward to see the equality $U_d = e^{iH_d}$, where $H_d$ is a real diagonal matrix: $H_d = diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We then have

$$U = V^\dagger U_d V = V^\dagger e^{iH_d} V = e^{iH} \tag{4.4}$$

with $H = V^\dagger H_d V$. Because $H_d$ is real and diagonal, the matrix $H$ is hermitian:

$$H^\dagger = \left(V^\dagger H_d V\right)^\dagger = V^\dagger H_d^\dagger V = H. \tag{4.5}$$

**(b)** From the matrix identity $e^{TrA} = \det(e^A)$, we have for $U = e^{iH}$

$$e^{iTrH} = \det\left(e^{iH}\right) = \det U. \tag{4.6}$$

Thus $\det U = 1$ implies that $TrH = 0$.

## 4.2 SU($n$) matrices

The $n \times n$ unitary matrices with unit determinant form the SU($n$) group.

**(a)** Show that it has $n^2 - 1$ independent group parameters.

**(b)** Show that the maximum number of mutually commuting matrices in an SU($n$) group is $(n-1)$. (This is the *rank* of the group.)

### Solution to Problem 4.2

**(a)** To count the number of independent group parameters, it is easier to do so through the generator matrix. From the previous problem, we have $U = e^{iH}$, where $H$ is an $n \times n$ traceless hermitian matrix. For a general hermitian matrix, the diagonal elements must be real, $H_{ii} = H_{ii}^*$. Because of the traceless condition, this corresponds to $(n-1)$ independent parameters. There are altogether $(n^2 - n)$ off-diagonal elements and thus $(n^2 - n)$ independent parameters because each complex element corresponds to two real parameters, yet this factor of two is cancelled by the hermitian conditions $H_{ij} = H_{ji}^*$. Consequently, we have a total of $(n - 1 + n^2 - n) = (n^2 - 1)$ independent parameters.

**(b)** From the discussion in Part (a) we already know that there are $n-1$ independent diagonal SU($n$) matrices, which obviously must be mutually commutative. On the other hand, if there were more than $n-1$ mutually commuting matrices, they could all be diagonalized simultaneously, thus yielding more than $n-1$ independent diagonal matrices. This is impossible for $n \times n$ traceless hermitian generating matrices.

## 4.3 Reality of SU(2) representations

This problem illustrates the special property of the SU(2) representations, their being equivalent to their complex conjugate representations.

**(a)** For every $2 \times 2$ unitary matrix $U$ with unit determinant, show that there exists a matrix $S$ which connects $U$ to its complex conjugate matrix $U^*$ through the similarity transformation

$$S^{-1} U S = U^*. \tag{4.7}$$

**(b)** Suppose $\psi_1$ and $\psi_2$ are the bases for the spin-$\frac{1}{2}$ representation of SU(2) having eigenvalues of $\pm\frac{1}{2}$ for the diagonal generator $T_3$,

$$T_3 \psi_1 = \tfrac{1}{2}\psi_1 \quad \text{and} \quad T_3 \psi_2 = -\tfrac{1}{2}\psi_2, \tag{4.8}$$

calculate the eigenvalues of $T_3$ operating on $\psi_1^*$ and $\psi_2^*$, respectively.

### Solution to Problem 4.3

**(a)** We will prove this by explicit construction. Problem 4.1 taught us that the unitary matrix $U$ can be expressed in terms of its generating matrix $U = \exp iH$.

Thus the matrix $S$, if it exists, must have the property of

$$S^{-1}HS = -H^* \tag{4.9}$$

so that $S^{-1}US = S^{-1}(\exp iH)S = U^* = \exp(-iH^*)$. The generating matrix $H$, being a $2 \times 2$ traceless hermitian matrix, can be expanded in terms of the Pauli matrices

$$H = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \tag{4.10}$$

with real coefficients of expansion $a$s. Since $\sigma_1$ and $\sigma_3$ are real, $\sigma_2$ imaginary, we have

$$H^* = a_1\sigma_1 - a_2\sigma_2 + a_3\sigma_3. \tag{4.11}$$

Equation (4.9) can be translated into relations between $S$ and Pauli matrices: $S^{-1}\sigma_1 S = -\sigma_1$, $S^{-1}\sigma_2 S = \sigma_2$, and $S^{-1}\sigma_3 S = -\sigma_3$. Namely, the matrix $S$ must commute with $\sigma_2$, and anticommute with $\sigma_1$ and $\sigma_3$. This can be satisfied with

$$S = c\sigma_2 \tag{4.12}$$

where $c$ is some arbitrary constant. If we choose $c = 1$, the matrix $S$ is unitary and hermitian; for $c = i$, $S$ is real.

**(b)** The statement '$\psi_1$ and $\psi_2$ are the bases for the spin-$\frac{1}{2}$ representation of SU(2)' means that under an SU(2) transformation $(i = 1, 2)$

$$\psi_i \to \psi_i' = U_{ij}\psi_j \quad \text{with} \quad U = \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}). \tag{4.13}$$

In matrix notation, this is $\psi' = U\psi$. The complex conjugate equation is then

$$\psi'^* = U^*\psi^* = (S^{-1}US)\psi^* \quad \text{or} \quad (S\psi'^*) = U(S\psi^*). \tag{4.14}$$

This means that $S\psi^*$ has the same transformation properties as $\psi$. Explicitly, with $S = i\sigma_2$, we have

$$S\psi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}. \tag{4.15}$$

To say that it has the same transformation properties as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{4.16}$$

means that, for example,

$$T_3\begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}. \tag{4.17}$$

Namely, the eigenvalues of the $T_3$ generators are

$$\begin{aligned} t_3(\psi_2^*) &= t_3(\psi_1) = \tfrac{1}{2} \\ t_3(\psi_1^*) &= t_3(\psi_2) = -\tfrac{1}{2}. \end{aligned} \tag{4.18}$$

**Remark.** This shows that the $T = \frac{1}{2}$ representation is equivalent to its complex conjugate representation. We say that it is a *real representation*. This property can be extended to all other representations of the SU(2) group, because all other representations can be obtained from the $T = \frac{1}{2}$ representation by tensor product. Part (b) shows that the matrix $S$ transforms any real diagonal matrix, e.g. $\sigma_3$, into the *negative* of itself. In other words, $S$ will transform any eigenvalue to its negative. Thus the existence of such a matrix $S$ requires that the eigenvalues of the hermitian-generating matrix occur in pairs of the form $\pm\alpha_1, \pm\alpha_2, \ldots$ (or are zero). It is then clear that for groups of SU($n$) with $n \geq 3$, such a matrix $S$ cannot exist as eigenvalues of higher-rank special unitary groups do not have such a special pairwise structure.

## 4.4 An identity for unitary matrices

**(a)** Show that if $A$ and $B$ are two $n \times n$ matrices, we have the Baker–Hausdorff relation

$$\begin{aligned} e^{iA}Be^{-iA} = B + i[A, B] + \frac{i^2}{2!}[A, [A, B]] + \cdots \\ + \frac{i^n}{n!}[A, [A, \ldots [A, B] \ldots]] + \cdots \end{aligned} \tag{4.19}$$

**(b)** Show that the matrix $B$ is invariant (up to a phase) under the transformations generated by the matrix $A$, if these two matrices satisfy the commutation relation of $[A, B] = B$.

**Solution to Problem 4.4**

**(a)** The matrix $J$, defined as $J(\lambda) \equiv e^{i\lambda A}Be^{-i\lambda A}$, being a function of some real parameter $\lambda$, can be differentiated to yield:

$$\begin{aligned}
\frac{dJ}{d\lambda} &= e^{i\lambda A}i[A, B]e^{-i\lambda A} \Rightarrow \left.\frac{dJ}{d\lambda}\right|_{\lambda=0} = i[A, B] \equiv iC_1 \\
\frac{d^2J}{d\lambda^2} &= e^{i\lambda A}i^2[A, [A, B]]e^{-i\lambda A} \Rightarrow \left.\frac{d^2J}{d\lambda^2}\right|_{\lambda=0} = i^2[A, [A, B]] \equiv i^2C_2 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\frac{d^nJ}{d\lambda^n} &= e^{i\lambda A}i^n[A, C_{n-1}]e^{-i\lambda A} \Rightarrow \left.\frac{d^nJ}{d\lambda^n}\right|_{\lambda=0} = i^n[A, C_{n-1}] \equiv i^nC_n.
\end{aligned}$$

Expand $J(\lambda)$ in a Taylor series:

$$J(\lambda) = \sum_{n=0}^{\infty} \left.\frac{d^nJ}{d\lambda^n}\right|_{\lambda=0} \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} i^nC_n\frac{\lambda^n}{n!} \tag{4.20}$$

where $C_0 = B$, $C_1 = [A, B]$, and $C_n = [A, C_{n-1}]$. Setting $\lambda = 1$, we have the desired result

$$e^{iA}Be^{-iA} = B + i[A, B] + \frac{i^2}{2!}[A, [A, B]] + \cdots . \tag{4.21}$$

**(b)** To show that 'the matrix $B$ is invariant (up to a phase) under transformations generated by matrix $A$' means to show that

$$e^{i\alpha A} B e^{-i\alpha A} = B \tag{4.22}$$

for an arbitrary real parameter $\alpha$. But from Part (a) we have already shown that

$$e^{i\alpha A} B e^{-i\alpha A} = \sum_{n=0}^{\infty} i^n C_n \frac{\alpha^n}{n!} \tag{4.23}$$

where $C_0 = B$, $C_1 = [A, B]$, and $C_n = [A, C_{n-1}]$. For the case at hand of $[A, B] = B$ we have $C_n = B$ for all $n = 0, 1, \ldots$

$$e^{i\alpha A} B e^{-i\alpha A} = B \sum_{n=0}^{\infty} i^n \frac{\alpha^n}{n!} = B e^{i\alpha}. \tag{4.24}$$

This is the claimed result.

## 4.5 An identity for SU(2) matrices

Prove the identity for $2 \times 2$ unitary matrices generated by Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$:

$$\exp(i\mathbf{r} \cdot \boldsymbol{\sigma}) = \cos r + (\hat{\mathbf{r}} \cdot \boldsymbol{\sigma}) \sin r \tag{4.25}$$

where $r = |\mathbf{r}|$ is the magnitude of the vector $\mathbf{r}$ and $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector.

### Solution to Problem 4.5

We will first derive a useful identity for Pauli matrices. Consider the multiplication of two matrices

$$\begin{aligned}
(\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) &= (\sigma_i \sigma_j) A_i B_j \\
&= \tfrac{1}{2}[(\sigma_i \sigma_j + \sigma_j \sigma_i) + (\sigma_i \sigma_j - \sigma_j \sigma_i)] A_i B_j \\
&= \tfrac{1}{2}(\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j]) A_i B_j \\
&= \tfrac{1}{2}(2\delta_{ij} + 2i\varepsilon_{ijk}\sigma_k) A_i B_j
\end{aligned} \tag{4.26}$$

where we have used the basic commutation relations satisfied by the Pauli matrices:

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}. \tag{4.27}$$

Thus we have the identity

$$(\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \times \mathbf{B}). \tag{4.28}$$

Set $\mathbf{A} = \mathbf{B} = \mathbf{r}$, we get $(\mathbf{r} \cdot \boldsymbol{\sigma})^2 = r^2 + i\boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{r}) = r^2$ and $(\mathbf{r} \cdot \boldsymbol{\sigma})^3 = r^2(\mathbf{r} \cdot \boldsymbol{\sigma}) = r^3(\hat{\mathbf{r}} \cdot \boldsymbol{\sigma})$. It is then straightforward to see that

$$(\mathbf{r} \cdot \boldsymbol{\sigma})^{2n} = r^{2n} \quad \text{and} \quad (\mathbf{r} \cdot \boldsymbol{\sigma})^{2n+1} = r^{2n+1}(\hat{\mathbf{r}} \cdot \boldsymbol{\sigma}) \tag{4.29}$$

with $n = 1, 2, \ldots$. The desired identity for the unitary matrix then follows as

$$\begin{aligned}\exp(i\mathbf{r}\cdot\boldsymbol{\sigma}) &= \sum_n \frac{i^n}{n!}(\mathbf{r}\cdot\boldsymbol{\sigma})^n \\ &= \sum_{n=\text{even}} \frac{i^n}{n!}r^n + (\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})\sum_{n=\text{odd}} \frac{i^n}{n!}r^n \\ &= \cos r + (\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})\sin r. \end{aligned} \tag{4.30}$$

**Remark.** This relation holds only for $2 \times 2$ unitary matrices and does not hold for higher-dimensional cases, where anticommutation relations are much more complicated than just the Kronecker delta.

## 4.6 SU(3) algebra in terms of quark fields

**(a)** Given a set of composite quark field operators

$$F^i \equiv \int q^\dagger(x)\frac{\lambda^i}{2}q(x)\, d^3x \tag{4.31}$$

where the quark field operators

$$q(x) = \begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{pmatrix} \tag{4.32}$$

satisfy the anticommutation relations

$$\left\{q_a(x), q_b^\dagger(y)\right\}\Big|_{x_0=y_0} = \delta_{ab}\,\delta^3(x-y), \tag{4.33}$$

and where $\lambda^i$, with $i = 1, 2, \ldots, 8$, are the Gell-Mann matrices

$$\left[\frac{\lambda^i}{2}, \frac{\lambda^j}{2}\right] = i f^{ijk}\frac{\lambda^k}{2}, \tag{4.34}$$

show that $\{F^i\}$, if assumed to be time-independent, generate the Lie algebra SU(3):

$$[F^i, F^j] = i f^{ijk} F^k. \tag{4.35}$$

**(b)** Calculate the commutators $[W_a^b, W_c^d]$ for the non-hermitian generators

$$W_a^b = \int q_b^\dagger(x) q_a(x)\, d^3x. \tag{4.36}$$

Show that $W_2^1$ is just the isospin raising operator. Similarly, $W_3^2$ and $W_3^1$ are, respectively, the U-spin and V-spin raising operators.

**Solution to Problem 4.6**

**(a)** The proof can be obtained by applying to the commutator $[F^i,\ F^j]$ the identity of

$$[AB, CD] = -AC\{D, B\} + A\{B, C\}D - C\{A, D\}B + \{C, A\}DB, \tag{4.37}$$

which for the present case has $\{A, C\} = \{B, D\} = 0$,

$$\begin{aligned}
[F^i, F^j] &= \int d^3x\, d^3y \left[ q_a^\dagger(x) \left(\frac{\lambda^i}{2}\right)_{ab} q_b(x),\ q_c^\dagger(y) \left(\frac{\lambda^j}{2}\right)_{cd} q_d(y) \right] \\
&= \int d^3x\, d^3y\, \delta^3(x-y) \Bigg[ q_a^\dagger(x) \left(\frac{\lambda^i}{2}\right)_{ab} \delta_{bc} \left(\frac{\lambda^j}{2}\right)_{cd} q_d(y) \\
&\quad - q_c^\dagger(y) \left(\frac{\lambda^j}{2}\right)_{cd} \delta_{ad} \left(\frac{\lambda^i}{2}\right)_{ab} q_b(x) \Bigg] \\
&= \int d^3x\, q^\dagger(x) \left[\frac{\lambda^i}{2}, \frac{\lambda^j}{2}\right] q(x) = i f^{ijk} F^k,
\end{aligned} \tag{4.38}$$

where, because $F^i$s are assumed to be time-independent, we have chosen $x_0 = y_0$ for convenience, and applied the equal-time anticommutator of the quark field operators.

**(b)** Again from the identity eqn (4.37) and the quark field anticommutation relations, we have

$$\begin{aligned}
\left[W_a^b, W_c^d\right] &= \int d^3x\, d^3y \left[ q_b^\dagger(x) q_a(x),\ q_d^\dagger(y) q_c(y) \right] \\
&= \int d^3x\, d^3y\, \delta^3(x-y) \left[ q_b^\dagger(x) \delta_a^d q_c(y) - q_d^\dagger(y) \delta_c^b q_a(x) \right] \\
&= \delta_a^d W_c^b - \delta_c^b W_a^d .
\end{aligned} \tag{4.39}$$

If we write $(q_1, q_2, q_3) \equiv (u, d, s)$, the non-hermitian operator

$$W_2^1 = \int q_1^\dagger(x) q_2(x)\, d^3x = \int u^\dagger(x)\, d(x)\, d^3x \tag{4.40}$$

is shown to be an operator which transforms a $d$-quark to a $u$. Clearly $W_2^1$ is the isospin raising operator. Similarly, we have $s \xrightarrow{W_3^2} d$ and $s \xrightarrow{W_3^1} u$.

**Remark.** In this notation the third component of the isospin generator $T_3$ takes the form of

$$\begin{aligned}
T_3 &= \tfrac{1}{2} \int (u^\dagger u - d^\dagger d)\, d^3x = \tfrac{1}{2} \int (q_1^\dagger q_1 - q_2^\dagger q_2)\, d^3x \\
&= \tfrac{1}{2}(W_1^1 - W_2^2)
\end{aligned} \tag{4.41}$$

and the hypercharge in the form of

$$Y = \tfrac{1}{3} \int (u^\dagger u + d^\dagger d - 2s^\dagger s)\, d^3x = \tfrac{1}{3}(W_1^1 + W_2^2 - 2W_3^3). \tag{4.42}$$

## 4.7 Combining two spin-$\frac{1}{2}$ states

Consider a doublet $\psi = (\psi_1, \psi_2)$ of the $SU(2)$ group. Show that the composite operators $S = \psi^\dagger\psi$ and $\mathbf{V} = \psi^\dagger \boldsymbol{\tau}\psi$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ being the usual Pauli matrices, transform as a scalar and a vector respectively. Also, demonstrate the vectorial transformation property of **V** in several ways:

**(a)** for a general infinitesimal rotation,

**(b)** for a finite rotation around the 3-axis,

**(c)** for a general finite rotation.

**Solution to Problem 4.7**

Under the SU(2) transformation, we have

$$\psi \longrightarrow \psi' = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}\psi, \quad \psi^\dagger \longrightarrow \psi'^\dagger = \psi^\dagger e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2} \tag{4.43}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are the three arbitrary real parameters. It is clear that $\psi^\dagger\psi$ is an invariant under SU(2) transformation.

$$S' = \psi'^\dagger\psi' = \psi^\dagger e^{+i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}\psi = \psi^\dagger\psi = S. \tag{4.44}$$

**(a)** To demonstrate the vectorial transformation property of **V** under a general infinitesimal rotation

$$\begin{aligned}\psi &\longrightarrow \psi' \simeq (1 - i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)\psi,\\ \psi^\dagger &\longrightarrow \psi'^\dagger \simeq \psi^\dagger(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2),\end{aligned} \tag{4.45}$$

we note that the transformed composite operator can be written

$$\begin{aligned}\mathbf{V}' &= \psi'^\dagger\boldsymbol{\tau}\psi' \simeq \psi^\dagger(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)\boldsymbol{\tau}(1 - i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)\psi\\ &= \psi^\dagger\left(\boldsymbol{\tau} + i\left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2},\ \boldsymbol{\tau}\right] + O(\alpha^2)\right)\psi.\end{aligned} \tag{4.46}$$

But the commutation relation for Pauli matrices yields

$$\left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2},\ \tau_k\right] = \alpha_j\left[\frac{\tau_j}{2},\ \tau_k\right] = i\alpha_j\epsilon_{jkl}\tau_l = i(\boldsymbol{\tau}\times\boldsymbol{\alpha})_k. \tag{4.47}$$

Thus we have demonstrated the vectorial nature of **V** under the infinitesimal SU(2) transformation

$$\mathbf{V}' = \psi^\dagger(\boldsymbol{\tau} - \boldsymbol{\tau}\times\boldsymbol{\alpha})\psi = \mathbf{V} - \mathbf{V}\times\boldsymbol{\alpha}. \tag{4.48}$$

**(b)** To demonstrate the vectorial transformation property of **V** under a finite rotation around the 3-axis:

$$\psi' = R\psi, \quad \psi'^\dagger = \psi^\dagger R^\dagger, \qquad \text{with} \quad R = e^{-i\alpha_3\tau_3/2}, \tag{4.49}$$

the transformed **V** operator can be written

$$\mathbf{V}' = \psi'^{\dagger}\boldsymbol{\tau}\psi' = \psi^{\dagger}R^{\dagger}\boldsymbol{\tau}R\psi = \psi^{\dagger}e^{i\alpha_3\tau_3/2}\,\boldsymbol{\tau}\,e^{-i\alpha_3\tau_3/2}\psi. \tag{4.50}$$

Applying the formula of eqn (4.19)

$$R^{\dagger}\boldsymbol{\tau}R = \boldsymbol{\tau} + i\left[\frac{\alpha_3\tau_3}{2},\ \boldsymbol{\tau}\right] + \frac{i^2}{2!}\left[\frac{\alpha_3\tau_3}{2},\ \left[\frac{\alpha_3\tau_3}{2},\ \boldsymbol{\tau}\right]\right] + \cdots \tag{4.51}$$

we clearly see that, because $[(\tau_3/2), \tau_3] = 0$, and thus $R^{\dagger}\tau_3 R = \tau_3$, the third component of **V** is unchanged under a rotation around the 3-axis. For the other two components we need to calculate $[(\tau_3/2), \tau_{1,2}]$. This can be considerably simplified if we work with the combinations $\tau_{\pm} = \tau_1 \pm i\tau_2$, which obey the commutation relation of

$$\left[\frac{\tau_3}{2},\ \tau_{\pm}\right] = \pm\tau_{\pm}. \tag{4.52}$$

In particular, we have

$$\left[\frac{\tau_3}{2}, \left[\frac{\tau_3}{2}, \cdots\left[\frac{\tau_3}{2},\ \tau_+\right]\cdots\right]\right] = \tau_+ \tag{4.53}$$

and thus

$$R^{\dagger}\tau_+ R = \tau_+\left(1 + (i\alpha_3) + \frac{1}{2!}(i\alpha_3)^2 + \cdots\right) = \tau_+\,e^{i\alpha_3}, \tag{4.54}$$

$$R^{\dagger}\tau_- R = \tau_-\left(1 + (-i\alpha_3) + \frac{1}{2!}(-i\alpha_3)^2 + \cdots\right) = \tau_-\,e^{-i\alpha_3}. \tag{4.55}$$

It then follows that

$$R^{\dagger}\tau_1 R = \frac{1}{2}\left(\tau_+\,e^{i\alpha_3} + \tau_-\,e^{-i\alpha_3}\right) = \cos\alpha_3\tau_1 - \sin\alpha_3\tau_2 \tag{4.56}$$

and similarly

$$R^{\dagger}\tau_2 R = \frac{1}{2}\left(\tau_+\,e^{-i\alpha_3} + \tau_-\,e^{+i\alpha_3}\right) = \sin\alpha_3\tau_1 + \cos\alpha_3\tau_2. \tag{4.57}$$

Consequently, the three components of **V** have the following transformation property under a finite transformation around the 3-axis:

$$\begin{aligned} V_1' &= \cos\alpha_3 V_1 - \sin\alpha_3 V_2, \\ V_2' &= \sin\alpha_3 V_1 + \cos\alpha_3 V_2, \\ V_3' &= V_3. \end{aligned} \tag{4.58}$$

**(c)** For the case of arbitrary SU(2) transformation, we have

$$\mathbf{V}' = \psi'^{\dagger}\boldsymbol{\tau}\psi' = \psi^{\dagger}U^{\dagger}\boldsymbol{\tau}U\psi \qquad \text{where} \quad U = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}. \tag{4.59}$$

$$U^{\dagger}\boldsymbol{\tau}U = \boldsymbol{\tau} + i\left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right),\ \boldsymbol{\tau}\right] + \frac{i^2}{2!}\left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right), \left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right),\ \boldsymbol{\tau}\right]\right] + \cdots.$$

The basic commutator can be calculated as follows:

$$\left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right),\ \tau_k\right] = \alpha_j\left[\frac{\tau_j}{2},\ \tau_k\right] = \alpha_j i\epsilon_{jkl}\tau_l = (-\boldsymbol{\alpha}\cdot\mathbf{t})_{kl}\tau_l \tag{4.60}$$

where we have recognized the spin-1 representation of the rotation operator

$$(t_j)_{kl} = -i\epsilon_{jkl}. \tag{4.61}$$

For the double commutator,

$$\begin{aligned}\left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right),\ \left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right),\ \tau_k\right]\right] &= (-\boldsymbol{\alpha}\cdot\mathbf{t})_{kl}\left[\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right),\ \tau_l\right]\\ &= (-\boldsymbol{\alpha}\cdot\mathbf{t})_{kl}(-\boldsymbol{\alpha}\cdot\mathbf{t})_{lj}\tau_j\\ &= (-\boldsymbol{\alpha}\cdot\mathbf{t})^2_{kl}\tau_l\end{aligned} \tag{4.62}$$

and so on. Thus

$$U^\dagger\tau_k U = \left[1 + (-i\boldsymbol{\alpha}\cdot\mathbf{t}) + \frac{1}{2!}(-i\boldsymbol{\alpha}\cdot\mathbf{t})^2 + \cdots\right]_{kl}\tau_l = (e^{-i\boldsymbol{\alpha}\cdot\mathbf{t}})_{kl}\tau_l, \tag{4.63}$$

or

$$V_j \longrightarrow V'_j = (e^{-i\boldsymbol{\alpha}\cdot\mathbf{t}})_{jk}V_k. \tag{4.64}$$

## 4.8 The SU(2) adjoint representation

**(a)** Suppose $\boldsymbol{\phi}$ transforms as a vector under SU(2) as discussed in eqn (4.64):

$$\phi_j \longrightarrow \phi'_j = (e^{-i\boldsymbol{\alpha}\cdot\mathbf{t}})_{jk}\phi_k. \tag{4.65}$$

Show that the transformation law for the $2\times 2$ matrix defined by $\hat{\Phi} = \boldsymbol{\tau}\cdot\boldsymbol{\phi}$ is given by

$$\hat{\Phi} \longrightarrow \hat{\Phi}' = U^\dagger\hat{\Phi}U \qquad \text{with} \quad U = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}. \tag{4.66}$$

**(b)** Suppose $\hat{\Sigma}$ is a $2\times 2$ hermitian traceless matrix which transforms as

$$\hat{\Sigma} \longrightarrow \hat{\Sigma}' = U^\dagger\hat{\Sigma}U \qquad \text{with} \quad U = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}. \tag{4.67}$$

Show that $\hat{\Sigma}'$ is also hermitian traceless, and with $\det\hat{\Sigma}' = \det\hat{\Sigma}$. Since $\hat{\Sigma}$ and $\hat{\Sigma}'$ are hermitian and traceless, they can be expanded in terms of Pauli matrices

$$\hat{\Sigma} = \boldsymbol{\tau}\cdot\boldsymbol{\phi} \qquad \text{and} \qquad \hat{\Sigma}' = \boldsymbol{\tau}\cdot\boldsymbol{\phi}'. \tag{4.68}$$

Show that $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$ are related by a rotation.

**(c)** Suppose we have the nucleon in the isodoublet representation $N = (p,\ n)$, and the pion in the isotriplet representation $\boldsymbol{\pi} = (\pi_1,\ \pi_2,\ \pi_3)$ with $\pi_3 = \pi^0$, $(\pi_1 - i\pi_2)/\sqrt{2} = \pi^+$, and $(\pi_1 + i\pi_2)/\sqrt{2} = \pi^-$, construct the SU(2) invariant pion–nucleon $\pi NN$ coupling.

**Solution to Problem 4.8**

**(a)** We will show that $\hat{\Phi}' = U^{\dagger}\hat{\Phi}U$ follows from the transformation eqn (4.65):

$$\hat{\Phi}' = \phi'_j \tau_j = (e^{-i\boldsymbol{\alpha}\cdot\mathbf{t}})_{jk}\phi_k\tau_j \tag{4.69}$$

$$= \phi_k\left[1 + (-i\boldsymbol{\alpha}\cdot\mathbf{t}) + \frac{1}{2!}(-i\boldsymbol{\alpha}\cdot\mathbf{t})^2 + \cdots\right]_{kl}\tau_l$$

which can be written, according to eqn (4.63), as

$$\hat{\Phi}' = \phi_k U^{\dagger}\tau_k U = U^{\dagger}\hat{\Phi}U. \tag{4.70}$$

**Remark.** This problem shows that there are two alternative ways to describe the transformation of the vector representation (more generally, the adjoint representation): as in eqn (4.64) or as in eqn (4.66).

**(b)** To show that $\hat{\Sigma}'$ is hermitian if $\hat{\Sigma}$ is hermitian:

$$\hat{\Sigma}'^{\dagger} = (U^{\dagger}\hat{\Sigma}U)^{\dagger} = U^{\dagger}\hat{\Sigma}^{\dagger}U = U^{\dagger}\hat{\Sigma}U = \hat{\Sigma}'. \tag{4.71}$$

To show that $\hat{\Sigma}'$ is traceless if $\Sigma$ is traceless:

$$tr\hat{\Sigma}' = trU^{\dagger}\hat{\Sigma}U = trUU^{\dagger}\hat{\Sigma} = tr\hat{\Sigma} = 0. \tag{4.72}$$

To show that $\det\hat{\Sigma}' = \det\hat{\Sigma}$:

$$\det\hat{\Sigma}' = \det U^{\dagger}\hat{\Sigma}U = \det UU^{\dagger}\hat{\Sigma} = \det\hat{\Sigma}. \tag{4.73}$$

Expanding $\hat{\Sigma}'$ and $\hat{\Sigma}$ in terms of Pauli matrices:

$$\hat{\Sigma} = \boldsymbol{\tau}\cdot\boldsymbol{\phi} = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix}, \tag{4.74}$$

it is easy to calculate their determinants:

$$\det\hat{\Sigma} = -\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right). \tag{4.75}$$

Thus the above result of $\det\hat{\Sigma}' = \det\hat{\Sigma}$ implies that the transformation $\boldsymbol{\phi} \longrightarrow \boldsymbol{\phi}'$ leaves the length $|\boldsymbol{\phi}|$ unchanged. This must be a rotation.

**(c)** From (a) and (b) we see that the $2\times 2$ hermitian traceless matrix $\hat{\Pi} = \boldsymbol{\tau}\cdot\boldsymbol{\pi}$, formed from an SU(2) vector $\boldsymbol{\pi}$, transforms by the similarity transformation:

$$\hat{\Pi} \longrightarrow \hat{\Pi}' = U^{\dagger}\hat{\Pi}U \qquad \text{with} \quad U = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2}. \tag{4.76}$$

In this form, it is easy to see that the product $N^{\dagger}\hat{\Pi}N$, where $N$ is an SU(2) doublet, is invariant under SU(2) transformations. This suggests the invariant pion–nucleon $\pi NN$ coupling to be

$$\mathcal{L}_{\pi NN} = g\bar{N}\boldsymbol{\tau}\cdot\boldsymbol{\pi}N = g(\bar{p},\,\bar{n})\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix}$$

$$= g\left[(\bar{p}p - \bar{n}n)\pi^0 + \sqrt{2}(\bar{p}n\pi^+ + \bar{n}p\pi^-)\right], \tag{4.77}$$

and thus the relations among coupling constants are

$$g_{\bar{p}p\pi^0} = -g_{\bar{n}n\pi^0} = \frac{1}{\sqrt{2}}g_{\bar{p}n\pi^+} = \frac{1}{\sqrt{2}}g_{\bar{n}p\pi^-} = g. \tag{4.78}$$

**Remark.** The relation $g_{\bar{p}n\pi^+} = g_{\bar{n}p\pi^-}$ follows from the hermiticity of the Lagrangian density (or charge conjugation).

## 4.9 Couplings of SU(2) vector representations

**(a)** The $\rho$ vector meson has isospin 1 (it has three charge states: $\rho^+$, $\rho^0$, $\rho^-$). Construct the SU(2) invariant $\rho\pi\pi$ coupling.

**(b)** The $\omega$ vector meson has isospin 0. Construct the SU(2) invariant $\omega\rho\pi$ coupling.

### Solution to Problem 4.9

**(a)** Since $\rho$ has spin 1 and isospin 1, we can represent the $\rho$-fields as $\boldsymbol{\rho}_\mu(x)$ and define a $2 \times 2$ matrix $\hat{P}_\mu = \boldsymbol{\tau} \cdot \boldsymbol{\rho}_\mu$. As one has seen in Problems 4.8 and 4.9, it transforms under SU(2) as

$$\hat{P}_\mu \longrightarrow \hat{P}'_\mu = U^\dagger \hat{P}_\mu U \tag{4.79}$$

just as the pion matrix $\hat{\Pi} \longrightarrow \hat{\Pi}' = U^\dagger \hat{\Pi} U$, where $U$ is an arbitrary $2 \times 2$ SU(2) matrix. This suggests the invariant coupling to be

$$\begin{aligned}
\mathcal{L}_{\rho\pi\pi} &= g tr(\hat{P}_\mu \partial^\mu \hat{\Pi} \hat{\Pi}) \\
&= g tr \begin{pmatrix} \rho^{0\mu} & \sqrt{2}\rho^{+\mu} \\ \sqrt{2}\rho^{-\mu} & -\rho^{0\mu} \end{pmatrix} \begin{pmatrix} \partial_\mu \pi^0 & \sqrt{2}\partial_\mu \pi^+ \\ \sqrt{2}\partial_\mu \pi^- & -\partial_\mu \pi^0 \end{pmatrix} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \\
&= g \left[ 2\rho^{0\mu} \left( \partial_\mu \pi^+ \pi^- - \partial_\mu \pi^- \pi^+ \right) + 2\rho^{+\mu} \left( \partial_\mu \pi^- \pi^0 - \partial_\mu \pi^0 \pi^- \right) \right. \\
&\quad \left. + 2\rho^{-\mu} \left( \partial_\mu \pi^+ \pi^0 - \partial_\mu \pi^0 \pi^+ \right) \right].
\end{aligned} \tag{4.80}$$

This implies, for example, the equality of decay rates

$$\Gamma(\rho^0 \to \pi^+\pi^-) = \Gamma(\rho^+ \to \pi^+\pi^0) = \Gamma(\rho^- \to \pi^-\pi^0). \tag{4.81}$$

**Remark 1.** The decay $\rho^0 \to \pi^0\pi^0$ is forbidden because the $(\pi^0\pi^0)$ system can only have even orbital angular momentum because of the Bose statistics. Hence angular momentum conservation will forbid this decay. Note: the same argument can be applied to the vector gauge boson $Z$ to forbid the decay into two identical Higgs (scalar or pseudoscalar) bosons.

**Remark 2.** The other possible coupling $tr(\hat{P}_\mu \hat{\Pi} \partial^\mu \hat{\Pi})$ is not independent of the one just considered. This can be seen by applying the Pauli matrix identity

$$(\boldsymbol{\tau} \cdot \mathbf{A})(\boldsymbol{\tau} \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B}) + i\, \boldsymbol{\tau} \cdot (\mathbf{A} \times \mathbf{B}) \tag{4.82}$$

which implies that

$$tr\left(\hat{P}_\mu \partial^\mu \hat{\Pi} \hat{\Pi}\right) = i\, \boldsymbol{\rho}_\mu (\partial^\mu \boldsymbol{\pi} \times \boldsymbol{\pi}), \tag{4.83}$$

$$tr\left(\hat{P}_\mu \hat{\Pi} \partial^\mu \hat{\Pi}\right) = -i\, \boldsymbol{\rho}_\mu (\partial^\mu \boldsymbol{\pi} \times \boldsymbol{\pi}). \tag{4.84}$$

**(b)** The SU(2) invariant Lorentz scalar combination out of the $\rho$, $\pi$, and $\omega$ meson fields can be constructed as

$$\begin{aligned}\mathcal{L}_{\rho\pi\omega} &= gtr(\hat{P}_\mu \hat{\Pi})\omega^\mu \\ &= 2g\left(\rho_\mu^+\pi^- + \rho_\mu^-\pi^+ + \rho_\mu^0\pi^0\right)\omega^\mu. \end{aligned} \tag{4.85}$$

## 4.10 Isospin breaking effects

Exact SU(2) symmetry implies the degeneracy for particles in the same irreducible representation of SU(2). But the SU(2) isospin symmetry is broken in nature by electromagnetism as well as by up-and-down quark mass difference. The first-order electromagnetic breaking, involving the emission and absorption of a photon (and thus the electromagnetic charge operator $Q$ acting twice), contains an isospin-changing $\Delta I = 1$ piece, as well as a $\Delta I = 2$ piece. On the other hand, the quark mass-difference:

$$m_u\bar{u}u + m_d\bar{d}d = \frac{m_u + m_d}{2}(\bar{u}u + \bar{d}d) + \frac{m_u - m_d}{2}(\bar{u}u - \bar{d}d) \tag{4.86}$$

contributes only a $\Delta I = 1$ breaking, as the last term transforms as the third component of the isospin generator $I_3$. Thus the strong interaction Hamiltonian can be written as

$$\mathcal{H} = \mathcal{H}_{(0)} + \mathcal{H}'_{(1)} + \mathcal{H}'_{(2)} \tag{4.87}$$

where $\mathcal{H}_{(0)}$ is SU(2) invariant, $\mathcal{H}'_{(2)}$ is the $\Delta I = 2$ electromagnetic breaking term, and the $\Delta I = 1$ piece $\mathcal{H}'_{(1)}$ contains both the electromagnetic and up–down mass-difference breakings. In this problem you are asked to calculate the first-order mass shifts due to $\mathcal{H}'$ by using the Wigner–Eckart theorem for the following isomultiplets:

**(a)** $I = \frac{1}{2}: (p, n)$,

**(b)** $I = 1: (\Sigma^+, \Sigma^0, \Sigma^-)$,

**(c)** $I = \frac{3}{2}: (\Delta^{++}, \Delta^+, \Delta^0, \Delta^-)$.

**Solution to Problem 4.10**

According to the Wigner–Eckart theorem, the matrix elements of a tensor operator $O_T^M$, having isospin $T$ and third component value $M$, have the simple structure of

$$\left\langle I', I'_3 \left| O_T^M \right| I, I_3 \right\rangle = \left\langle I', I'_3 \right| T, M; I, I_3\big\rangle \left\langle I' \left\| O_T \right\| I \right\rangle \tag{4.88}$$

where the first factor on the right-hand side is the Clebsch–Gordon coefficient and the second factor is the reduced matrix element, which is independent of $I_3$, $I'_3$, and $M$. For this problem, the operator $O = \mathcal{H}'_{(1)}$ and $\mathcal{H}'_{(2)}$ which transforms as the $(T = 1, M = 0)$ and $(T = 2, M = 0)$, respectively.

**(a)** $I = \frac{1}{2}$ **multiplet** $(p, n)$. The first-order mass shift due to $\mathcal{H}'_{(1)}$ can be evaluated by the Wigner–Eckart theorem as

$$\begin{aligned}\delta^{(1)} m_p &= \left\langle p \left| \mathcal{H}'_{(1)} \right| p \right\rangle = \left\langle \tfrac{1}{2}, \tfrac{1}{2} \left| \mathcal{H}'_{(1)} \right| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle \\ &= \left\langle \tfrac{1}{2}, \tfrac{1}{2} \right| 1, 0; \tfrac{1}{2}, \tfrac{1}{2} \right\rangle \delta^{(1)} m_N = -\sqrt{\tfrac{1}{3}} \delta^{(1)} m_N \end{aligned} \tag{4.89}$$

and

$$\begin{aligned}\delta^{(1)} m_n &= \left\langle n \left| \mathcal{H}'_{(1)} \right| n \right\rangle = \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \left| \mathcal{H}'_{(1)} \right| \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \\ &= \left\langle \tfrac{1}{2}, -\tfrac{1}{2} \right| 1, 0; \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \delta^{(1)} m_N = \sqrt{\tfrac{1}{3}} \delta^{(1)} m_N \end{aligned} \tag{4.90}$$

where $\delta^{(1)} m_N$ is the $I_3$-independent reduced matrix element.

Exactly the same calculation shows that the $\Delta I = 2$ shifts $\delta^{(2)} m_p = \delta^{(2)} m_n = 0$ as the corresponding Clebsch–Gordon coefficients vanish (because an $I = 2$ operator cannot connect two $I = 1/2$ states).

In this way we find that the proton and neutron mass shifts $\delta m_{p,n} = \delta^{(1)} m_{p,n}$ have the same magnitude but are opposite in sign:

$$\delta m_p = -\delta m_n. \tag{4.91}$$

**Remark 1.** We can apply this result to any other $I = \frac{1}{2}$ multiplets. For example,

$$\delta m_{\Xi^0} = -\delta m_{\Xi^-}, \quad \delta m_{K^0} = -\delta m_{K^-}, \text{ etc.} \tag{4.92}$$

**Remark 2.** Alternatively, we can write down an effective mass term in the Lagrangian, which contains an operator having an isospin value of $(T = 1,\ M = 0)$. For the isodoublets this can be represented by a $2 \times 2$ matrix, $\tau_3$. The effective mass term for the nucleon can then be written as

$$\begin{aligned}\mathcal{L}_{\delta m_N} &= \bar{N} \delta m_N \tau_3 N = \delta m_N (\bar{p},\ \bar{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} \\ &= \delta m_N\, (\bar{p} p - \bar{n} n), \end{aligned} \tag{4.93}$$

which yields

$$\delta m_p = -\delta m_n = \delta m_N. \tag{4.94}$$

**(b)** $I = 1$ **multiplet** $(\Sigma^+,\ \Sigma^0, \Sigma^-)$ The Wigner–Eckart theorem yields

$$\begin{aligned}\delta^{(1)} m_{\Sigma^+} &= \left\langle \Sigma^+ \left| \mathcal{H}'_{(1)} \right| \Sigma^+ \right\rangle = \left\langle 1, +1 \left| \mathcal{H}'_{(1)} \right| 1, +1 \right\rangle \\ &= \langle 1, +1 | 1, 0; 1, +1 \rangle \delta^{(1)} m_\Sigma = \sqrt{\tfrac{1}{2}} \delta^{(1)} m_\Sigma \end{aligned} \tag{4.95}$$

$$\begin{aligned}\delta^{(1)} m_{\Sigma^0} &= \left\langle \Sigma^0 \left| \mathcal{H}'_{(1)} \right| \Sigma^0 \right\rangle = \left\langle 1, 0 \left| \mathcal{H}'_{(1)} \right| 1, 0 \right\rangle \\ &= \langle 1, 0 | 1, 0; 1, 0 \rangle \delta^{(1)} m_\Sigma = 0 \end{aligned} \tag{4.96}$$

$$\delta^{(1)}m_{\Sigma^-} = \left\langle \Sigma^- \left| \mathcal{H}'_{(1)} \right| \Sigma^- \right\rangle = \left\langle 1, -1 \left| \mathcal{H}'_{(1)} \right| 1, -1 \right\rangle$$
$$= \langle 1, -1 | 1, 0; 1, -1 \rangle \delta^{(1)} m_\Sigma = -\sqrt{\tfrac{1}{2}}\delta^{(1)} m_\Sigma \qquad (4.97)$$

and

$$\delta^{(2)} m_{\Sigma^+} = \left\langle \Sigma^+ \left| \mathcal{H}'_{(2)} \right| \Sigma^+ \right\rangle = \langle 1, +1 | 2, 0; 1, +1 \rangle \delta^{(2)} m_\Sigma = \sqrt{\tfrac{1}{10}} \delta^{(2)} m_\Sigma \qquad (4.98)$$

$$\delta^{(2)} m_{\Sigma^0} = \left\langle \Sigma^0 \left| \mathcal{H}'_{(2)} \right| \Sigma^0 \right\rangle = \langle 1, 0 | 2, 0; 1, 0 \rangle \delta^{(2)} m_\Sigma = -\sqrt{\tfrac{2}{5}} \delta^{(2)} m_\Sigma \qquad (4.99)$$

$$\delta^{(2)} m_{\Sigma^-} = \left\langle \Sigma^- \left| \mathcal{H}'_{(2)} \right| \Sigma^- \right\rangle = \langle 1, -1 | 2, 0; 1, -1 \rangle \delta^{(2)} m_\Sigma = \sqrt{\tfrac{1}{10}} \delta^{(2)} m_\Sigma. \qquad (4.100)$$

Combining these results and using the notation $m_1 = \sqrt{\frac{1}{2}}\delta^{(1)} m_\Sigma$ and $m_2 = \sqrt{\frac{1}{10}}\delta^{(2)} m_\Sigma$, we have

$$\begin{aligned} m_{\Sigma^+} &= m_0 + m_1 + m_2, \\ m_{\Sigma^0} &= m_0 + \ 0\ - 2m_2, \\ m_{\Sigma^-} &= m_0 - m_1 + m_2. \end{aligned} \qquad (4.101)$$

The $I = 1$ and $I = 2$ mass splittings can then be isolated:

$$m_1 = \tfrac{1}{2}(m_{\Sigma^+} - m_{\Sigma^-}), \qquad (4.102)$$

$$m_2 = \tfrac{1}{4}(m_{\Sigma^+} + m_{\Sigma^-} - 2m_{\Sigma^0}). \qquad (4.103)$$

While $m_1$ contains both the electromagnetic and up–down mass difference effects, $m_2$ is purely electromagnetic in origin.

**Remark.** The same analysis holds for the isotriplet pions. In particular, we have, besides $\delta^{(1)} m_{\pi^0} = 0$, the result

$$\delta^{(1)} m_{\pi^+} = -\delta^{(1)} m_{\pi^-}. \qquad (4.104)$$

But $\pi^+$ is the antiparticle of $\pi^-$ and should have the same mass. Hence

$$\delta m_{\pi^+} = \delta m_{\pi^-}. \qquad (4.105)$$

The only way to reconcile these two eqns (4.104) and (4.105) is to have the reduced matrix element $\delta^{(1)} m_\pi = 0$. (The Wigner–Eckart theorem does not by itself give any information about the reduced matrix element.) The same result can be seen from writing out the mass term in the Lagrangian:

$$\begin{aligned} \mathcal{L}_m(\pi) &= \Delta^{(1)} m_\pi \, tr \left( \hat{\Pi} \tau_3 \hat{\Pi} \right) \\ &= \Delta^{(1)} m_\pi \, tr \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \\ &= \Delta^{(1)} m_\pi (\pi^{02} - 2\pi^+\pi^- - \pi^{02} + 2\pi^+\pi^-) = 0. \end{aligned} \qquad (4.106)$$

Thus the pion mass differences (and for $\rho$ mesons also) are entirely due to the $I = 2$ electromagnetic corrections.

**(c)** $I = 3/2$ **multiplet** ($\Delta^{++},\ \Delta^{+},\ \Delta^{0},\ \Delta^{-}$)
The Wigner–Eckart theorem yields

$$\begin{aligned}
\delta^{(1)} m_{\Delta^{++}} &= \left\langle \tfrac{3}{2}, +\tfrac{3}{2} \right| 1, 0;\ \tfrac{3}{2}, +\tfrac{3}{2} \Big\rangle \delta^{(1)} m_\Delta = \sqrt{\tfrac{3}{5}}\delta^{(1)} m_\Delta \\
\delta^{(1)} m_{\Delta^{+}} &= \left\langle \tfrac{3}{2}, +\tfrac{1}{2} \right| 1, 0;\ \tfrac{3}{2}, +\tfrac{1}{2} \Big\rangle \delta^{(1)} m_\Delta = \sqrt{\tfrac{1}{15}}\delta^{(1)} m_\Delta \\
\delta^{(1)} m_{\Delta^{0}} &= \left\langle \tfrac{3}{2}, -\tfrac{1}{2} \right| 1, 0;\ \tfrac{3}{2}, -\tfrac{1}{2} \Big\rangle \delta^{(1)} m_\Delta = -\sqrt{\tfrac{1}{15}}\delta^{(1)} m_\Delta \\
\delta^{(1)} m_{\Delta^{-}} &= \left\langle \tfrac{3}{2}, -\tfrac{3}{2} \right| 1, 0;\ \tfrac{3}{2}, -\tfrac{3}{2} \Big\rangle \delta^{(1)} m_\Delta = -\sqrt{\tfrac{3}{5}}\delta^{(1)} m_\Delta
\end{aligned} \tag{4.107}$$

and

$$\begin{aligned}
\delta^{(2)} m_{\Delta^{++}} &= \left\langle \tfrac{3}{2}, +\tfrac{3}{2} \right| 2, 0;\ \tfrac{3}{2}, +\tfrac{3}{2} \Big\rangle \delta^{(2)} m_\Delta = \sqrt{\tfrac{1}{5}}\delta^{(2)} m_\Delta \\
\delta^{(2)} m_{\Delta^{+}} &= \left\langle \tfrac{3}{2}, +\tfrac{1}{2} \right| 2, 0;\ \tfrac{3}{2}, +\tfrac{1}{2} \Big\rangle \delta^{(2)} m_\Delta = -\sqrt{\tfrac{1}{5}}\delta^{(2)} m_\Delta \\
\delta^{(2)} m_{\Delta^{0}} &= \left\langle \tfrac{3}{2}, -\tfrac{1}{2} \right| 2, 0;\ \tfrac{3}{2}, -\tfrac{1}{2} \Big\rangle \delta^{(2)} m_\Delta = -\sqrt{\tfrac{1}{5}}\delta^{(2)} m_\Delta \\
\delta^{(2)} m_{\Delta^{-}} &= \left\langle \tfrac{3}{2}, -\tfrac{3}{2} \right| 2, 0;\ \tfrac{3}{2}, -\tfrac{3}{2} \Big\rangle \delta^{(2)} m_\Delta = \sqrt{\tfrac{1}{5}}\delta^{(2)} m_\Delta.
\end{aligned} \tag{4.108}$$

Combining these results we have

$$\begin{aligned}
m_{\Delta^{++}} &= m_0 + 3m_1 + m_2, \\
m_{\Delta^{+}} &= m_0 + m_1 - m_2, \\
m_{\Delta^{0}} &= m_0 - m_1 - m_2, \\
m_{\Delta^{-}} &= m_0 - 3m_1 + m_2.
\end{aligned} \tag{4.109}$$

Besides allowing us to isolate the $I = 1$ and $I = 2$ mass splittings, these equations also imply a mass relation of

$$m_{\Delta^{++}} - m_{\Delta^{-}} = 3(m_{\Delta^{+}} - m_{\Delta^{0}}). \tag{4.110}$$

This simply reflects the absence of an $I = 3$ piece in the symmetry-breaking Hamiltonian.

## 4.11 Spin wave function of three quarks

As an exercise in Clebsch–Gordon coefficient calculation, construct the spin states of baryons, which are composed of three spin-$\frac{1}{2}$ quarks.

### Solution to Problem 4.11

The possible spin states for two quarks are $S_{12} = 0, 1$, where $\mathbf{S}_{12} = \mathbf{S}_1 + \mathbf{S}_2$. As discussed in the text, the $S_{12} = 1$ states $|S_{12},\ S_{12,z}\rangle$ are

$$\begin{aligned}
|1, 1\rangle &= \alpha_1\alpha_2 \\
|1, 0\rangle &= \tfrac{1}{\sqrt{2}}(\alpha_1\beta_2 + \beta_1\alpha_2) \\
|1, -1\rangle &= \beta_1\beta_2
\end{aligned} \tag{4.111}$$

where $\alpha = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and $\beta = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ are the spin-up and spin-down states, respectively. Also we have the spin-zero combination:

$$|0, 0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2). \tag{4.112}$$

Combining the $S_{12} = 1$ states with $S_3 = \frac{1}{2}$, we obtain $S = \frac{3}{2}$ and $\frac{1}{2}$ states, where $\mathbf{S} = \mathbf{S}_{12} + \mathbf{S}_3 = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$.

$$\left|S = \tfrac{3}{2}, S_z = \tfrac{3}{2}\right\rangle = |1, 1\rangle \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \alpha_1\alpha_2\alpha_3. \tag{4.113}$$

To reach the other $S = \frac{3}{2}$ states, we use the lowering operators

$$S_-|S, S_z\rangle = \left[(S + S_z)(S - S_z + 1)\right]^{1/2} |S, S_z - 1\rangle \tag{4.114}$$

to obtain

$$S_- \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = \sqrt{3} \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle. \tag{4.115}$$

On the other hand, $S_- = (S_{12})_- + (S_3)_-$. Thus

$$\begin{aligned} S_-|1, 1\rangle \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle &= (S_{12})_-|1, 1\rangle \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle + |1, 1\rangle(S_3)_- \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \\ &= \sqrt{2}|1, 0\rangle\alpha_3 + |1, 1\rangle\beta_3. \end{aligned} \tag{4.116}$$

Combining eqns (4.115), (4.116), and (4.111), we get

$$\begin{aligned} \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}\left[\sqrt{2}|1, 0\rangle\alpha_3 + |1, 1\rangle\beta_3\right] \\ &= \frac{1}{\sqrt{3}}[(\alpha_1\beta_2 + \beta_1\alpha_2)\alpha_3 + \alpha_1\alpha_2\beta_3] \\ &= \frac{1}{\sqrt{3}}[\alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\alpha_2\beta_3]. \end{aligned} \tag{4.117}$$

Similar to eqn (4.113), we have

$$\left|S = \tfrac{3}{2}, S_z = -\tfrac{3}{2}\right\rangle = |1, -1\rangle \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = \beta_1\beta_2\beta_3, \tag{4.118}$$

and using $S_+$ we can obtain

$$\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[\alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3]. \tag{4.119}$$

The state $\left|S = \frac{1}{2}, S_z = \frac{1}{2}\right\rangle$ must be orthogonal to $\left|S = \frac{3}{2}, S_z = \frac{1}{2}\right\rangle$ in eqn ( 4.117):

$$\begin{aligned} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_S &= \frac{1}{\sqrt{3}}\left[-|1, 0\rangle\alpha_3 + \sqrt{2}|1, 1\rangle\beta_3\right] \\ &= \frac{1}{\sqrt{6}}[2\alpha_1\alpha_2\beta_3 - (\beta_1\alpha_2 + \alpha_1\beta_2)\alpha_3], \end{aligned} \tag{4.120}$$

where the subscript $S$ signifies the symmetry property of the state under the permutation of quarks $1 \leftrightarrow 2$. Similarly, we have

$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_S = \frac{1}{\sqrt{6}}[2\beta_1\beta_2\alpha_3 - (\alpha_1\beta_2 + \beta_1\alpha_2)\beta_3]. \tag{4.121}$$

But $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$ can also be obtained from combining the $S_{12} = 0$ and $S_3 = \frac{1}{2}$ states. Such combinations are antisymmetric under the permutation of quarks $1 \leftrightarrow 2$:

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)\alpha_3$$
$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)\beta_3. \tag{4.122}$$

To summarize, we have four $S = \frac{3}{2}$ states, which are completely symmetric under any permutation of all three quarks (1, 2, 3):

$$\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = \alpha_1\alpha_2\alpha_3$$
$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[\alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\alpha_2\beta_3]$$
$$\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[\alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3]$$
$$\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \beta_1\beta_2\beta_3 \tag{4.123}$$

and two $S = \frac{1}{2}$ states $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle_S \equiv \chi_{M,S}$, which have mixed symmetry with respect to the permutation of (1, 2, 3) but are symmetric under the permutation of $1 \leftrightarrow 2$:

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_S = \frac{1}{\sqrt{6}}[2\alpha_1\alpha_2\beta_3 - (\beta_1\alpha_2 + \alpha_1\beta_2)\alpha_3]$$
$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_S = \frac{1}{\sqrt{6}}[2\beta_1\beta_2\alpha_3 - (\alpha_1\beta_2 + \beta_1\alpha_2)\beta_3] \tag{4.124}$$

and two $S = \frac{1}{2}$ states $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle_A \equiv \chi_{M,A}$, which have mixed symmetry with respect to the permutation of (1, 2, 3) but are antisymmetric under the permutation of $1 \leftrightarrow 2$:

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)\alpha_3$$
$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)\beta_3. \tag{4.125}$$

**Remark.** If we are interested in the isospin of three non-strange light quarks $u$ and $d$ (as for the nucleons and the $\Delta$ resonances), we can work out the corresponding isospin wave functions by the simple substitution of $\alpha \to u$ and $\beta \to d$:

**(a)** Symmetric isospin $I = \frac{3}{2}$ states

$$\Delta^{++} = \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = u_1u_2u_3$$
$$\Delta^{+} = \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[u_1d_2u_3 + d_1u_2u_3 + u_1u_2d_3]$$
$$\Delta^{0} = \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[u_1d_2d_3 + d_1u_2d_3 + d_1d_2u_3]$$
$$\Delta^{-} = \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = d_1d_2d_3. \tag{4.126}$$

**(b)** Mixed-symmetric (but symmetric with respect to the interchange $1 \leftrightarrow 2$) $I = \frac{1}{2}$ states $\phi_{M,S}$:

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_S = \frac{1}{\sqrt{6}}[2u_1u_2d_3 - (d_1u_2 + u_1d_2)u_3]$$
$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_S = \frac{1}{\sqrt{6}}[2d_1d_2u_3 - (u_1d_2 + d_1u_2)d_3]. \tag{4.127}$$

**(c)** Mixed-symmetric (but antisymmetric with respect to the interchange $1 \leftrightarrow 2$) $I = \frac{1}{2}$ states $\phi_{M,A}$:

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(u_1 d_2 - d_1 u_2) u_3$$
$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(u_1 d_2 - d_1 u_2) d_3. \tag{4.128}$$

## 4.12 Permutation symmetry in the spin-isospin space

Show that the spin and isospin combination of the mixed symmetry states discussed in Problem 14.11, $\chi_{M,S}\phi_{M,S} + \chi_{M,A}\phi_{M,A}$ is invariant under the general permutations of particles indices. *Hint:* Such permutation operations are represented by orthogonal matrices in the 2D spaces spanned by mixed-symmetric spin wave functions, $\chi_1 = \chi_{M,S}$ and $\chi_2 = \chi_{M,A}$, or isospin wave functions, $\phi_1 = \phi_{M,S}$ and $\phi_2 = \phi_{M,A}$.

### Solution to Problem 4.12

The general permutation of three indices can be denoted as

$$P = \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \tag{4.129}$$

where $(1, 2, 3)$ are replaced by $(i_1, i_2, i_3)$—thus a permutation of the particle indices $(1, 2, 3)$. There are six elements in this permutation group $S_3$:

$$P_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad P_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$P_{123} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad P_{132} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

It is clear that under any of the permutation operations, the mixed-symmetric spin wave functions $\chi_{M,S}$ and $\chi_{M,A}$ transform into linear combinations of $\chi_{M,S}$ and $\chi_{M,A}$. The trivial examples are $I$ and $P_{12}$. By construction, we have

$$P_{12}\chi_{M,S} = \chi_{M,S}, \qquad P_{12}\chi_{M,A} = -\chi_{M,A}, \tag{4.130}$$

i.e. $\chi_{M,S}$ and $\chi_{M,A}$ are eigenstates of $P_{12}$ and $I$. The operator $I$ and $P_{12}$ can be written as an orthogonal matrices:

$$\hat{I}\hat{\chi} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \hat{P}_{12}\hat{\chi} \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{4.131}$$

The more general cases can be exemplified by $P_{13}$ acting on

$$\chi_1 = \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle_S = \frac{1}{\sqrt{6}}[2\alpha_1\alpha_2\beta_3 - (\beta_1\alpha_2 + \alpha_1\beta_2)\alpha_3],$$
$$\chi_2 = \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle_A = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)\alpha_3, \tag{4.132}$$

to yield

$$P_{13}\chi_1 = \frac{1}{\sqrt{6}}[2\alpha_3\alpha_2\beta_1 - (\beta_3\alpha_2 + \alpha_3\beta_2)\alpha_1]$$
$$= -\tfrac{1}{2}\chi_1 - \tfrac{\sqrt{3}}{2}\chi_2, \tag{4.133}$$

and

$$P_{13}\chi_2 = \frac{1}{\sqrt{2}}(\alpha_3\beta_2 - \beta_3\alpha_2)\alpha_1$$
$$= -\tfrac{\sqrt{3}}{2}\chi_1 + \tfrac{1}{2}\chi_2. \tag{4.134}$$

Thus $P_{13}$ can also be represented by an orthogonal matrix

$$\hat{P}_{13}\hat{\chi} \doteq \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{4.135}$$

It is not difficult to convince oneself that, because the permuted states $P\chi \to \chi'$ must remain orthogonal to each other, all the six permutation elements can be represented by orthogonal matrices:

$$\hat{P}^{\mathsf{T}}\hat{P} = \hat{P}\hat{P}^{\mathsf{T}} = \hat{I}. \tag{4.136}$$

From this property it follows that the combinations such as

$$\hat{\chi}^{\mathsf{T}}\hat{\chi} = \chi_1^2 + \chi_2^2, \qquad \hat{\chi}^{\mathsf{T}}\hat{\phi} = \chi_1\phi_1 + \chi_2\phi_2 \tag{4.137}$$

are invariant under all the permutations of the particle indices.

**Remark.** Since the combination $\chi_{M,S}\phi_{M,S} + \chi_{M,A}\phi_{M,A}$ is invariant under any permutation of quark indices, it is totally symmetric in the spin and isospin space. From this we can conclude that the nucleon wave function, which is a product of spin, isospin, and the totally antisymmetric colour wave functions, is totally antisymmetric with respect to interchange of any of its three quarks. This is compatible with the requirement of the (generalized) Pauli principle as the nucleon is a system of fermions (quarks).

## 4.13 Combining two fundamental representations

Work out the tensor products of the defining representations of SU(2) and SU(3):

**(a)** Let $\psi = \binom{\psi_1}{\psi_2} = \binom{u}{d}$ be an isospin doublet with its hermitian conjugate being $\psi^\dagger = \left(\psi_1^* \ \psi_2^*\right) = \left(u^\dagger \ d^\dagger\right)$. Find the isospin of the product $\psi_i^*\psi_j$ (where $i = 1, 2$).

**(b)** Let $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ be an SU(3) triplet. Decompose the product $\psi_i^*\psi_j$ (where $i = 1, 2, 3$) into irreducible representations of SU(3).

**Solution to Problem 4.13**

**(a)** It is useful to denote the complex conjugate $\psi_i^*$ by $\psi^i$. Namely, $\psi^i \equiv \psi_i^*$. From the fact that under an SU(2) transformation,

$$\psi_i \longrightarrow \psi_i' = U_{ij}\psi_j = U_i^j\psi_j \tag{4.138}$$

where $U_i^j \equiv U_{ij}$ and $U$ is unitary, we have

$$\psi_i^* \longrightarrow \psi_i'^* = U_{ij}^*\psi_j^* = \psi_j^*(U_i^j)^* \tag{4.139}$$

or

$$\psi^i \longrightarrow \psi^{i\prime} = \psi^j U_j^i \tag{4.140}$$

where $U_j^i \equiv (U_i^j)^*$. It is clear that the combination $\psi^i\psi_i$ is an SU(2) invariant:

$$\psi^{i\prime}\psi_i' = \psi^j U_j^i U_i^k \psi_k = \psi^j U_{ij}^* U_{ik}\psi_k = \psi^j \delta_j^k \psi_k = \psi^j\psi_j. \tag{4.141}$$

Thus $\psi^i\psi_i$ has isospin $I = 0$. It is easy to see that the remaining three combinations in the product $\psi^i\psi_j$ transform as an $I = 1$ triplet. We can remove the $I = 0$ combination by the following subtraction:

$$T_j^i = \psi^i\psi_j - \tfrac{1}{2}\delta_j^i(\psi^k\psi_k) \tag{4.142}$$

which has the property of $T_i^i = 0$. The $T_j^i$ components can be explicitly written out to be:

$$T_2^1 = \psi^1\psi_2 = u^\dagger d \sim \pi^-$$
$$T_1^2 = \psi^2\psi_1 = d^\dagger u \sim \pi^+$$

and

$$\begin{aligned} T_1^1 &= \psi^1\psi_1 - \tfrac{1}{2}\left(\psi^1\psi_1 + \psi^2\psi_2\right) = \tfrac{1}{2}\left(\psi^1\psi_1 - \psi^2\psi_2\right) \\ &= \tfrac{1}{2}(u^\dagger u - d^\dagger d) \sim \tfrac{1}{\sqrt{2}}\pi^0 \end{aligned} \tag{4.143}$$

$$\begin{aligned} T_2^2 &= \psi^2\psi_2 - \tfrac{1}{2}\left(\psi^1\psi_1 + \psi^2\psi_2\right) = -\tfrac{1}{2}\left(\psi^1\psi_1 - \psi^2\psi_2\right) \\ &= -\tfrac{1}{2}(u^\dagger u - d^\dagger d) \sim \tfrac{-1}{\sqrt{2}}\pi^0. \end{aligned} \tag{4.144}$$

Sometimes it is convenient to write $T_j^i$ as a traceless matrix

$$\hat{T} = \begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}. \tag{4.145}$$

We can summarize the result in the form of a direct sum:

$$\mathbf{2}^* \times \mathbf{2} = \mathbf{1} + \mathbf{3}, \tag{4.146}$$

where the representations are denoted by their respective dimensions. The triplet is called the adjoint representation of SU(2).

**(b)** Again the SU(3) invariant trace

$$\psi^i \psi_i = u^\dagger u + d^\dagger d + s^\dagger s \tag{4.147}$$

is an SU(3) singlet. The remaining eight components transform as the octet representation under SU(3),

$$\mathbf{3} \times \mathbf{3}^* = \mathbf{1} + \mathbf{8}. \tag{4.148}$$

Following the same procedure as in (a) we can display the adjoint representation of SU(3) as

$$A^i_j = \psi^i \psi_j - \tfrac{1}{3}\delta^i_j (\psi^k \psi_k). \tag{4.149}$$

To display the quantum numbers of various components:

$$\begin{aligned}
A^1_2 &= u^\dagger d \sim \pi^-, \qquad & A^2_1 &= d^\dagger u \sim \pi^+, \\
A^1_3 &= u^\dagger s \sim K^-, \qquad & A^3_1 &= s^\dagger u \sim K^+, \\
A^3_2 &= s^\dagger d \sim K^0, \qquad & A^2_3 &= d^\dagger s \sim \overline{K^0},
\end{aligned} \tag{4.150}$$

for the diagonal elements

$$A^1_1 = u^\dagger u - \frac{1}{3}(u^\dagger u + d^\dagger d + s^\dagger s) \sim \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} \tag{4.151}$$

where

$$\pi^0 = \frac{1}{\sqrt{2}}(u^\dagger u - d^\dagger d), \quad \eta^0 = \frac{1}{\sqrt{6}}(u^\dagger u + d^\dagger d - 2s^\dagger s). \tag{4.152}$$

Similarly, we can work out

$$A^2_2 \sim -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} \qquad \text{and} \qquad A^3_3 \sim -\frac{2\eta^0}{\sqrt{6}}. \tag{4.153}$$

These octet components can be organized as a traceless hermitian matrix:

$$\hat{A} = \begin{pmatrix} A^1_1 & A^2_1 & A^3_1 \\ A^1_2 & A^2_2 & A^3_2 \\ A^1_3 & A^2_3 & A^3_3 \end{pmatrix} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta^0}{\sqrt{6}} \end{pmatrix}.$$

Because of the transformation properties of the defining representation and its conjugate:

$$\psi_i \longrightarrow \psi_i{}' = U^j_i \psi_j, \qquad \psi^i \longrightarrow \psi^{i\prime} = \psi^j U^i_j, \tag{4.154}$$

the adjoint representation transforms as

$$A^i_j \longrightarrow A'^i_j = U^i_k U^l_j A^k_l = (U_{ik})^* A^k_l (U_{lj}), \tag{4.155}$$

or, in terms of matrix multiplication, it has the simple form of

$$\hat{A} \longrightarrow \hat{A}' = \hat{U}^\dagger \hat{A} \hat{U}. \tag{4.156}$$

Let us recall that here the $U$ matrix is the defining representation of the SU(3) group.

## 4.14 SU(3) invariant octet baryon–meson couplings

The baryon octet can be represented by a $3 \times 3$ matrix

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & \frac{-2\Lambda}{\sqrt{6}} \end{pmatrix} \quad \overline{B} = \begin{pmatrix} \frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Sigma^-} & \overline{\Xi^-} \\ \overline{\Sigma^+} & -\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Xi^0} \\ \overline{p} & \overline{n} & \frac{-2\overline{\Lambda}}{\sqrt{6}} \end{pmatrix}$$

and the pseudoscalar meson octet by

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^8}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & \frac{-2\eta^8}{\sqrt{6}} \end{pmatrix}.$$

**(a)** Construct the SU(3) invariant $\bar{B}BM$ couplings.

**(b)** Express all the above meson–baryon couplings in terms of two SU(3) symmetric couplings. This implies numerous coupling relations. But most of them actually follow from SU(2) isospin symmetry. Thus it is useful to express these SU(2) invariant couplings directly in terms of the two SU(3) couplings.

**Solution to Problem 4.14**

**(a)** In terms of these baryon and meson matrices, it is clear that there are only two invariant $\bar{B}BM$ couplings as there are only two independent traces of multiplying these matrices together:

$$\begin{aligned} \mathcal{L}_{\bar{B}BM} &= \sqrt{2}[g_1\, tr(\bar{B}BM) + g_2\, tr(\bar{B}MB)] \\ &= \sqrt{2}\left[g_1\left(\bar{B}_i^j B_j^k M_k^i\right) + g_2\left(\bar{B}_i^j M_j^k B_k^i\right)\right]. \end{aligned} \tag{4.157}$$

A common way to express these couplings is to write

$$g_1 = \frac{D+F}{2}, \qquad g_2 = \frac{D-F}{2}, \tag{4.158}$$

so that

$$\mathcal{L}_{\bar{B}BM} = \frac{F}{\sqrt{2}} tr(\bar{B}[B, M]) + \frac{D}{\sqrt{2}} tr(\bar{B}\{M, B\}). \tag{4.159}$$

Namely, the $F$ coupling is proportional to the commutator, and $D$ to the anticommutator. We now work out these couplings in terms of individual baryon and meson fields:

**(i) The $\pi^+$ couplings**

$$\begin{aligned}
\mathcal{L}_{\pi^+} &= \sqrt{2}M_1^2\left(g_1\bar{B}_2^j B_j^1 + g_2\bar{B}_i^1 B_2^i\right)\\
&= \sqrt{2}\pi^+\Bigg\{g_1\left[\overline{\Sigma^+}\left(\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right)+\left(-\frac{\overline{\Sigma^0}}{\sqrt{2}}+\frac{\overline{\Lambda}}{\sqrt{6}}\right)\Sigma^- + \overline{\Xi^0}\,\Xi^-\right]\\
&\quad + g_2\left[\left(\frac{\overline{\Sigma^0}}{\sqrt{2}}+\frac{\overline{\Lambda}}{\sqrt{6}}\right)\Sigma^- + \overline{\Sigma^+}\left(-\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right)+\bar{p}n\right]\Bigg\}\\
&= \sqrt{2}\pi^+\Bigg[\frac{g_1-g_2}{\sqrt{2}}\left(\overline{\Sigma^+}\Sigma^0-\overline{\Sigma^0}\Sigma^-\right)+\frac{g_1+g_2}{\sqrt{6}}\left(\overline{\Sigma^+}\Lambda-\overline{\Lambda}\Sigma^-\right)\\
&\quad + g_1\overline{\Xi^0}\,\Xi^- + g_2\overline{p}n\Bigg].
\end{aligned}\tag{4.160}$$

Or

$$\begin{aligned}
\mathcal{L}_{\pi^+} = \sqrt{2}\pi^+\Bigg\{&\frac{F}{\sqrt{2}}\left(\overline{\Sigma^+}\Sigma^0-\overline{\Sigma^0}\Sigma^-\right)+\frac{D}{\sqrt{6}}\left(\overline{\Sigma^+}\Lambda-\overline{\Lambda}\Sigma^-\right)\\
&+\frac{D+F}{2}\,\overline{\Xi^0}\,\Xi^- + \frac{D-F}{2}\bar{p}n\Bigg\}.
\end{aligned}\tag{4.161}$$

**(ii) The $\pi^0$ coupling**

$$\begin{aligned}
\mathcal{L}_{\pi^0} &= \sqrt{2}\left[M_1^1\left(g_1\bar{B}_1^j B_j^1 + g_2\bar{B}_i^1 B_1^i\right)+M_2^2\left(g_1\bar{B}_2^j B_j^2 + g_2\bar{B}_i^2 B_2^i\right)\right]\\
&= \pi^0\Bigg\{g_1\Bigg[\left(\frac{\overline{\Sigma^0}}{\sqrt{2}}+\frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right)+\overline{\Sigma^-}\Sigma^- + \overline{\Xi^-}\,\Xi^-\\
&\quad - \overline{\Sigma^+}\Sigma^+ - \left(-\frac{\overline{\Sigma^0}}{\sqrt{2}}+\frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(-\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right)-\overline{\Xi^0}\,\Xi^0\Bigg]\\
&\quad + g_2\Bigg[\left(\frac{\overline{\Sigma^0}}{\sqrt{2}}+\frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right)+\overline{\Sigma^+}\Sigma^+ + \bar{p}p\\
&\quad - \overline{\Sigma^-}\Sigma^- - \left(-\frac{\overline{\Sigma^0}}{\sqrt{2}}+\frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(-\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right)-\bar{n}n\Bigg]\Bigg\}+\cdots\\
&= \pi^0\Bigg[(g_1-g_2)\left(\overline{\Sigma^-}\Sigma^- - \overline{\Sigma^+}\Sigma\right)+\frac{g_1+g_2}{\sqrt{3}}\left(\overline{\Sigma^0}\Lambda+\overline{\Lambda}\Sigma^0\right)\\
&\quad + g_1\left(\overline{\Xi^-}\,\Xi^- - \overline{\Xi^0}\,\Xi^0\right)+g_2(\bar{p}p-\bar{n}n)\Bigg].
\end{aligned}\tag{4.162}$$

Or

$$\mathcal{L}_{\pi^0} = \pi^0 \Bigg[ F \left( \overline{\Sigma^-}\Sigma^- - \overline{\Sigma^+}\Sigma^+ \right) + \frac{D}{\sqrt{3}} \left( \overline{\Sigma^0}\Lambda + \overline{\Lambda}\Sigma^0 \right)$$
$$+ \frac{D+F}{2} \left( \overline{\Xi^-}\Xi^- - \overline{\Xi^0}\Xi^0 \right) + \frac{D-F}{2} (\bar{p}p - \bar{n}n) \Bigg]. \quad (4.163)$$

**(iii) The $K^+$ couplings**

$$\begin{aligned}
\mathcal{L}_{K^+} &= \sqrt{2} M_1^3 \left( g_1 \bar{B}_3^j B_j^1 + g_2 \bar{B}_i^1 B_3^i \right) \\
&= \sqrt{2} K^+ \Bigg\{ g_1 \left[ \overline{p} \left( \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} \right) + \bar{n}\Sigma^- - \frac{2}{\sqrt{6}} \overline{\Lambda}\Xi^- \right] \\
&\quad + g_2 \left[ \left( \frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} \right) \Xi^- + \overline{\Sigma^+}\Xi^0 - \frac{2}{\sqrt{6}} \overline{p}\Lambda \right] \Bigg\} \\
&= \sqrt{2} K^+ \Bigg[ g_1 \left( \frac{\overline{p}\Sigma^0}{\sqrt{2}} + \bar{n}\Sigma^- \right) + g_2 \left( \frac{\overline{\Sigma^0}\Xi^-}{\sqrt{2}} + \overline{\Sigma^+}\Xi^0 \right) \\
&\quad + \frac{-2g_1 + g_2}{\sqrt{6}} \overline{\Lambda}\Xi^- + \frac{g_1 - 2g_2}{\sqrt{6}} \overline{p}\Lambda \Bigg]. \qquad (4.164)
\end{aligned}$$

Or

$$\mathcal{L}_{K^+} = \sqrt{2} K^+ \Bigg[ \frac{D+F}{2} \left( \frac{\overline{p}\Sigma^0}{\sqrt{2}} + \bar{n}\Sigma^- \right) + \frac{D-F}{2} \left( \frac{\overline{\Sigma^0}\Xi^-}{\sqrt{2}} + \overline{\Sigma^+}\Xi^0 \right)$$
$$- \frac{D+3F}{2\sqrt{6}} \overline{\Lambda}\Xi^- - \frac{D-3F}{2\sqrt{6}} \overline{p}\Lambda \Bigg]. \quad (4.165)$$

**(iv) The $K^0$ couplings**

$$\begin{aligned}
\mathcal{L}_{K^0} &= \sqrt{2} M_2^3 \left( g_1 \bar{B}_3^j B_j^2 + g_2 \bar{B}_i^2 B_3^i \right) \\
&= \sqrt{2} K^0 \Bigg\{ g_1 \left[ \overline{p}\Sigma^+ + \bar{n} \left( -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} \right) - \frac{2}{\sqrt{6}} \overline{\Lambda}\Xi^0 \right] \\
&\quad + g_2 \left[ \overline{\Sigma^-}\Xi^- + \left( -\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} \right) \Xi^0 - \frac{2}{\sqrt{6}} \overline{n}\Lambda \right] \Bigg\} \\
&= \sqrt{2} K^0 \Bigg[ g_1 \left( -\frac{\overline{n}\Sigma^0}{\sqrt{2}} + \bar{p}\Sigma^+ \right) + g_2 \left( -\frac{\overline{\Sigma^0}\Xi^0}{\sqrt{2}} + \overline{\Sigma^-}\Xi^- \right) \\
&\quad + \frac{-2g_1 + g_2}{\sqrt{6}} \overline{\Lambda}\Xi^0 + \frac{g_1 - 2g_2}{\sqrt{6}} \overline{n}\Lambda \Bigg]. \qquad (4.166)
\end{aligned}$$

Or

$$\mathcal{L}_{K^0} = \sqrt{2}K^0\Bigg[\frac{D+F}{2}\left(-\frac{\overline{n}\Sigma^0}{\sqrt{2}} + \bar{p}\Sigma^+\right) + \frac{D-F}{2}\left(-\frac{\overline{\Sigma^0}\Xi^0}{\sqrt{2}} + \overline{\Sigma^-}\Xi^-\right) - \frac{D+3F}{2\sqrt{6}}\,\overline{\Lambda}\Xi^0 - \frac{D-3F}{2\sqrt{6}}\,\overline{n}\Lambda\Bigg]. \tag{4.167}$$

**(v) The $\eta^8$ coupling**

$$\begin{aligned}\mathcal{L}_{\eta^8} &= \sqrt{2}\Big[M_1^1\left(g_1\bar{B}_1^jB_j^1 + g_2\bar{B}_i^1B_1^i\right) + M_2^2\left(g_1\bar{B}_2^jB_j^2 + g_2\bar{B}_i^2B_2^i\right) \\ &\quad + M_3^3\left(g_1\bar{B}_3^jB_j^3 + g_2\bar{B}_i^3B_3^i\right)\Big] \\ &= \frac{\eta^8}{\sqrt{3}}\Big[g_1\left(\bar{B}_1^jB_j^1 + \bar{B}_2^jB_j^2 - 2\bar{B}_3^jB_j^3\right) \\ &\quad + g_2\left(\bar{B}_i^1B_1^i + \bar{B}_i^2B_2^i - 2\bar{B}_i^3B_3^i\right)\Big] + \cdots\end{aligned}$$

so that

$$\begin{aligned}\mathcal{L}_{\eta^8} = \frac{\eta^8}{\sqrt{3}}\Bigg\{&g_1\Bigg[\left(\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}\right) + \overline{\Sigma^-}\Sigma^- + \overline{\Xi^-}\Xi^- + \overline{\Sigma^+}\Sigma^+ \\ &+ \left(-\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(-\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}\right) + \overline{\Xi^0}\Xi^0 - 2\bar{p}p - 2\bar{n}n - \frac{4}{3}\overline{\Lambda}\Lambda\Bigg] \\ &+ g_2\Bigg[\left(\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}\right) + \overline{\Sigma^+}\Sigma^+ + \overline{p}p + \overline{\Sigma^-}\Sigma^- \\ &+ \left(-\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}}\right)\left(-\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}\right) \\ &+ \overline{n}n - 2\overline{\Xi^-}\Xi^- - 2\overline{\Xi^0}\Xi^0 - \frac{4}{3}\overline{\Lambda}\Lambda\Bigg]\Bigg\} + \cdots\end{aligned}$$

Or

$$\begin{aligned}\mathcal{L}_{\eta^8} &= \frac{\eta^8}{\sqrt{3}}\Big[(g_1+g_2)\left(\overline{\Sigma^-}\Sigma^- + \overline{\Sigma^0}\Sigma^0 + \overline{\Sigma^+}\Sigma^+\right) - (g_1+g_2)\overline{\Lambda}\Lambda \\ &\quad + (g_1 - 2g_2)\left(\overline{\Xi^-}\Xi^- + \overline{\Xi^0}\Xi^0\right) - (2g_1 - g_2)(\overline{p}p + \overline{n}n)\Big] \\ &= \frac{\eta^8}{\sqrt{3}}\Big[D\left(\overline{\Sigma^-}\Sigma^- + \overline{\Sigma^0}\Sigma^0 + \overline{\Sigma^+}\Sigma^+\right) - D\overline{\Lambda}\Lambda \\ &\quad + \frac{3F-D}{2}\left(\overline{\Xi^-}\Xi^- + \overline{\Xi^0}\Xi^0\right) - \frac{D+3F}{2}(\overline{p}p + \overline{n}n)\Big].\end{aligned} \tag{4.168}$$

The other couplings are related by hermitian conjugation:

$$\mathcal{L}^\dagger_{\pi^+} = \mathcal{L}_{\pi^-}, \qquad \mathcal{L}^\dagger_{K^+} = \mathcal{L}_{K^-}, \qquad \mathcal{L}^\dagger_{K^0} = \mathcal{L}_{K^0}. \tag{4.169}$$

**(b)** The baryon octet contains the following isospin multiplets:

$$\begin{aligned}
\text{singlet}: &\quad \Lambda \\
\text{doublets}: &\quad N = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \Xi = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix} \\
\text{triplet}: &\quad \hat{\Sigma} = \begin{pmatrix} \Sigma^0 & \sqrt{2}\Sigma^+ \\ \sqrt{2}\Sigma^- & -\Sigma^0 \end{pmatrix}
\end{aligned} \tag{4.170}$$

while the meson octet contains

$$\begin{aligned}
\text{singlet}: &\quad \eta^8 \\
\text{doublets}: &\quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad K^c = \begin{pmatrix} \overline{K^0} \\ K^- \end{pmatrix} \\
\text{triplet}: &\quad \hat{\Pi} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.
\end{aligned} \tag{4.171}$$

From this it is straightforward to construct all the SU(2) invariant couplings, cf. Problem 4.8(b):

**(i) Three singlets**

$$\eta\bar{\Lambda}\Lambda \tag{4.172}$$

**(ii) One singlet ⊕ two doublets**

$$\eta\overline{N}N, \quad \eta\overline{\Xi}\Xi, \quad K\overline{\Lambda}\Xi, \quad K^c\overline{\Lambda}N \tag{4.173}$$

**(iii) One singlet ⊕ two triplets**

$$\eta\, tr\hat{\Sigma}^c\hat{\Sigma}, \quad \overline{\Lambda}\, tr\hat{\Sigma}\hat{\Pi} \tag{4.174}$$

**(iv) One triplet ⊕ two doublets**

$$\overline{N}\hat{\Pi}N, \quad \overline{\Xi}\hat{\Pi}\Xi, \quad \overline{N}\hat{\Sigma}K, \quad \overline{\Xi}\hat{\Sigma}K^c \tag{4.175}$$

**(v) Three triplets**

$$tr\,\hat{\Sigma}^c\hat{\Sigma}\hat{\Pi} \tag{4.176}$$

By working out the components such as

$$\begin{aligned}
\overline{N}\hat{\Pi}N &= (\bar{p}p - \bar{n}n)\pi^0 + \sqrt{2}(\bar{p}n\pi^+ + \bar{n}p\pi^-) \\
tr\hat{\Sigma}^c\hat{\Sigma}\hat{\Pi} &= \pi^+\left(\overline{\Sigma^+}\Sigma^0 - \overline{\Sigma^0}\Sigma^-\right) - \pi^-\left(\overline{\Sigma^-}\Sigma^0 - \overline{\Sigma^0}\Sigma^+\right) \\
&\quad + \pi^0\left(\overline{\Sigma^-}\Sigma^- - \overline{\Sigma^+}\Sigma^+\right)
\end{aligned} \tag{4.177}$$

and comparing them with the couplings shown above, we can easily express the SU(2) invariant couplings in terms of the SU(3) $D$ and $F$ couplings.

$$g(\pi NN) = \frac{D-F}{2}, \quad g(\pi\Sigma\Sigma) = F, \tag{4.178}$$

and similarly

$$\begin{aligned}
-g(\eta\Lambda\Lambda) &= g(\eta\Sigma\Sigma) = g(\pi\Lambda\Sigma) = \frac{D}{\sqrt{3}} \\
g(\eta NN) &= g(K\Lambda\Xi) = -\frac{D+3F}{2\sqrt{3}} \\
g(\eta\Xi\Xi) &= g(K\Lambda N) = -\frac{D-3F}{2\sqrt{3}} \\
g(\pi\Xi\Xi) &= -g(KN\Sigma) = -\frac{D+F}{2} \\
g(K\Xi\Sigma) &= -g(\pi NN) = -\frac{D-F}{2}.
\end{aligned} \tag{4.179}$$

**Remark.** Just as we have seen in Problem 8 that, in terms of the Cartesian components $\pi = (\pi_1\ \pi_2\ \pi_3)$, the coupling $\overline{N}\hat{\Pi}N$ can be written as $\overline{N}\boldsymbol{\tau}N\cdot\boldsymbol{\pi}$, the three triplet coupling $tr\,\hat{\Sigma}^c\hat{\Sigma}\hat{\Pi}$ can be written as $i\,\boldsymbol{\Sigma}^c\times\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}$, where we have used the identity

$$tr(\tau_l\tau_m\tau_n) = i\epsilon_{lmn}. \tag{4.180}$$

## 4.15 Isospin wave functions of two pions

Two pions can have total isospin $I = 2, 1, 0$. Use the relations

$$I_-|I, I_3\rangle = [(I+I_3)(I-I_3+1)]^{1/2}\,|I, I_3-1\rangle \tag{4.181}$$

$$I_+|I, I_3\rangle = [(I-I_3)(I+I_3+1)]^{1/2}\,|I, I_3+1\rangle \tag{4.182}$$

to construct the total isospin wave functions of two pions.

### Solution to Problem 4.15

**(a) The $I = 2$ states**
Starting from

$$|2, 2\rangle = |\pi_1^+\pi_2^+\rangle \tag{4.183}$$

we can use $I_-|2, 2\rangle = 2|2, 1\rangle$ and $I_-|1, 1\rangle = \sqrt{2}|1, 0\rangle$ or $I_-|\pi^+\rangle = \sqrt{2}|\pi^0\rangle$ to get

$$|2, 1\rangle = \frac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^0\right\rangle + \left|\pi_1^0\pi_2^+\right\rangle\right); \tag{4.184}$$

and use $I_-|2,1\rangle = \sqrt{6}|2,0\rangle$ and $I_-|\pi^0\rangle = \sqrt{2}|\pi^-\rangle$ to get

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left(\left|\pi_1^+\pi_2^-\right\rangle + 2\left|\pi_1^0\pi_2^0\right\rangle + \left|\pi_1^-\pi_2^+\right\rangle\right). \tag{4.185}$$

Similarly,

$$\begin{aligned}|2,-1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\pi_1^0\pi_2^-\right\rangle + \left|\pi_1^-\pi_2^0\right\rangle\right)\\ |2,-2\rangle &= \left|\pi_1^-\pi_2^-\right\rangle.\end{aligned} \tag{4.186}$$

**(b) The $I = 1$ states**

The $|1,1\rangle$ state is a linear combination of $\left|\pi_1^0\pi_2^-\right\rangle$ and $\left|\pi_1^-\pi_2^0\right\rangle$ which is orthogonal to the $|2,+1\rangle$ state. Thus if we write

$$|1,1\rangle = a\left|\pi_1^+\pi_2^0\right\rangle + b\left|\pi_1^0\pi_2^+\right\rangle \tag{4.187}$$

with $|a|^2 + |b|^2 = 1$, the orthogonality condition becomes $1/\sqrt{2}(a+b) = 0$. The solution of $a = -b = 1/\sqrt{2}$ can be chosen:

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^0\right\rangle - \left|\pi_1^0\pi_2^+\right\rangle\right). \tag{4.188}$$

This choice (as opposed to $a = -b = -1/\sqrt{2}$) corresponds to a particular convention for the Clebsch–Gordon coefficients. It is easy to see that using the isospin lowering operator $I_-$ we can get the other $I = 1$ states:

$$\begin{aligned}|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^-\right\rangle - \left|\pi_1^-\pi_2^+\right\rangle\right)\\ |1,-1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\pi_1^0\pi_2^-\right\rangle - \left|\pi_1^-\pi_2^0\right\rangle\right).\end{aligned} \tag{4.189}$$

**(c) The $I = 0$ state**

The $|0,0\rangle$ state must be orthogonal to both $|2,0\rangle$ and $|1,0\rangle$. This fixes it to be

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(\left|\pi_1^+\pi_2^-\right\rangle - \left|\pi_1^0\pi_2^0\right\rangle + \left|\pi_1^-\pi_2^+\right\rangle\right). \tag{4.190}$$

**Remark.** We note that the $I = 2$ and $I = 0$ states are symmetric under the interchange of particles $1 \leftrightarrow 2$, while the $I = 1$ states are antisymmetric. (This is why the combination $\left|\pi_1^0\pi_2^0\right\rangle$ is absent in the $|1,0\rangle$ state.) In general, for two particles with the same isospin, the largest total isospin states are symmetric under the interchange of particles $1 \leftrightarrow 2$. Then the next isospin states are antisymmetric, followed by symmetric states, etc. For example, in a system with two $I = 3/2$ particles, the state with $I = 3, 1$ is symmetric, while the one with $I = 2, 0$ is antisymmetric.

## 4.16 Isospins in non-leptonic weak processes

The low-energy $\Delta S = 1$ non-leptonic weak Hamiltonian is given by

$$\mathcal{H}_w = \frac{G_F}{\sqrt{2}}[\bar{u}\gamma^\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)u] + h.c. \tag{4.191}$$

The first term $(\bar{u}d)$ is an isospin state $|1, 1\rangle$ and the second term $(\bar{s}u)$ a $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$. From isospin addition of

$$|1, 1\rangle \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle, \tag{4.192}$$

we see that this weak Hamiltonian can be decomposed into two pieces with definite isospins:

$$\mathcal{H}_w = \mathcal{H}_{1/2} + \mathcal{H}_{3/2}. \tag{4.193}$$

**(a)** Use this isospin decomposition of $\mathcal{H}_w$ and Clebsch–Gordon coefficients to evaluate the decay amplitudes for

$$K^+ \to \pi^+\pi^0, \quad K^0 \to \pi^+\pi^-, \quad K^0 \to \pi^0\pi^0, \tag{4.194}$$

in terms of two reduced matrix elements.

**(b)** Repeat the same calculation for the decays $\Lambda \to p\pi^-$ and $\Lambda \to n\pi^0$.

### Solution to Problem 4.16

**(a)** Here we need to evaluate the matrix elements of $\langle\pi\pi|\mathcal{H}_w|K\rangle = \langle\pi\pi|\mathcal{H}_{1/2}|K\rangle + \langle\pi\pi|\mathcal{H}_{3/2}|K\rangle$. Since $(K^+, K^0)$ is an isospin doublet, $|K^+\rangle = \left|\frac{1}{2}, +\frac{1}{2}\right\rangle$ and $|K^0\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$, we have

$$\begin{aligned}
\mathcal{H}_{1/2}|K^+\rangle &= \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = |1, 1\rangle \\
\mathcal{H}_{3/2}|K^+\rangle &= \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \tfrac{\sqrt{3}}{2}|2, 1\rangle - \tfrac{1}{2}|1, 1\rangle,
\end{aligned} \tag{4.195}$$

$$\begin{aligned}
\mathcal{H}_{1/2}|K^0\rangle &= \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{2}}(|1, 0\rangle + |0, 0\rangle) \\
\mathcal{H}_{3/2}|K^0\rangle &= \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{2}}(|2, 0\rangle + |1, 0\rangle).
\end{aligned} \tag{4.196}$$

From the result obtained in Problem 4.15,

$$\begin{aligned}
|2, 1\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^0\right\rangle + \left|\pi_1^0\pi_2^+\right\rangle\right) \\
|1, 1\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^0\right\rangle - \left|\pi_1^0\pi_2^+\right\rangle\right),
\end{aligned} \tag{4.197}$$

we see that the final state of the decay $K^+ \to \pi^+\pi^0$ must be either of the isospin states $|2, 1\rangle$ or $|1, 1\rangle$. But from angular momentum conservation, $\pi^+\pi^0$ must be in a relative orbital angular momentum $L = 0$ state, which is symmetric under the interchange of $\pi^+ \leftrightarrow \pi^0$. Thus we must use the symmetric combination as in $|2, +1\rangle$ and only the $I = 3/2$ of the Hamiltonian can contribute:

$$\langle \pi^+\pi^0|\mathcal{H}_w|K^+\rangle = \langle \pi^+\pi^0(I=2)|\mathcal{H}_{3/2}|K^+\rangle = \left\langle 2, 1|\mathcal{H}_{3/2}|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle. \tag{4.198}$$

The Wigner–Eckart theorem can then be used to relate it to a reduced matrix element $\mathcal{A}_{3/2}$

$$\left\langle 2, 1 \left|\mathcal{H}_{3/2}\right| \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \left\langle 2, 1 \left| \tfrac{3}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right.\right\rangle \mathcal{A}_{3/2} = \tfrac{\sqrt{3}}{2}\mathcal{A}_{3/2} \tag{4.199}$$

Namely, the decay amplitude is evaluated in terms of the reduced matrix element as

$$T(K^+ \to \pi^+\pi^0) = \tfrac{\sqrt{3}}{2}\mathcal{A}_{3/2}. \tag{4.200}$$

As for the decays $K^0 \to \pi^+\pi^-$ and $K^0 \to \pi^0\pi^0$, we have from Problem 4.15

$$\begin{aligned}
|2, 0\rangle &= \tfrac{1}{\sqrt{6}}\left(\left|\pi_1^+\pi_2^-\right\rangle + 2\left|\pi_1^0\pi_2^0\right\rangle + \left|\pi_1^-\pi_2^+\right\rangle\right) \\
|1, 0\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^-\right\rangle - \left|\pi_1^-\pi_2^+\right\rangle\right) \\
|0, 0\rangle &= \tfrac{1}{\sqrt{3}}\left(\left|\pi_1^+\pi_2^-\right\rangle - \left|\pi_1^0\pi_2^0\right\rangle + \left|\pi_1^-\pi_2^+\right\rangle\right)
\end{aligned} \tag{4.201}$$

which shows that the $I = 1$ state is antisymmetric in $\pi^+ \leftrightarrow \pi^-$, which is forbidden by angular momentum conservation and Bose statistics. We then have

$$\begin{aligned}
\tfrac{1}{\sqrt{2}}\left(\left|\pi_1^+\pi_2^-\right\rangle + \left|\pi_1^-\pi_2^+\right\rangle\right) &= \sqrt{\tfrac{1}{3}}|2, 0\rangle + \sqrt{\tfrac{2}{3}}|0, 0\rangle \\
\left|\pi_1^0\pi_2^0\right\rangle &= \sqrt{\tfrac{2}{3}}|2, 0\rangle - \sqrt{\tfrac{1}{3}}|0, 0\rangle.
\end{aligned} \tag{4.202}$$

Thus

$$\begin{aligned}
\langle \pi^+\pi^-|\mathcal{H}_w|K^0\rangle &= \tfrac{1}{\sqrt{3}}(\langle 2, 0| + \sqrt{2}\langle 0, 0|)\mathcal{H}_w|K^0\rangle \\
&= \tfrac{1}{\sqrt{3}}\left(\langle 2, 0|\mathcal{H}_{3/2}|K^0\rangle + \sqrt{2}\langle 0, 0|\mathcal{H}_{1/2}|K^0\rangle\right) \\
&= \tfrac{1}{\sqrt{3}}\left[\left\langle 2, 0 \left| \tfrac{3}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right.\right\rangle \mathcal{A}_{3/2}.\right. \\
&\quad \left. + \sqrt{2}\left\langle 0, 0 \left| \tfrac{3}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right.\right\rangle \mathcal{A}_{1/2}\right] \\
&= \tfrac{1}{\sqrt{3}}\left(\tfrac{1}{\sqrt{2}}\mathcal{A}_{3/2} - \mathcal{A}_{1/2}\right) \\
&= \tfrac{1}{\sqrt{6}}\mathcal{A}_{3/2} - \tfrac{1}{\sqrt{3}}\mathcal{A}_{1/2}.
\end{aligned} \tag{4.203}$$

Or

$$T(K^0 \to \pi^+\pi^-) = \tfrac{1}{\sqrt{6}}\mathcal{A}_{3/2} - \tfrac{1}{\sqrt{3}}\mathcal{A}_{1/2}, \tag{4.204}$$

and similarly,

$$T(K^0 \to \pi^0\pi^0) = \tfrac{1}{\sqrt{3}}\mathcal{A}_{3/2} - \tfrac{1}{\sqrt{6}}\mathcal{A}_{1/2}. \tag{4.205}$$

**Remark.** Experimentally, it has been observed that

$$T(K^+ \to \pi^+\pi^0) \ll T(K^0 \to \pi^+\pi^-) \tag{4.206}$$

and

$$\frac{T(K^0 \to \pi^+\pi^-)}{T(K^0 \to \pi^0\pi^0)} \simeq \sqrt{2}. \tag{4.207}$$

This can then be translated through eqns (**??**), (**??**), and (**??**) into the amplitude relation of

$$\mathcal{A}_{3/2} \ll \mathcal{A}_{1/2}. \tag{4.208}$$

This is the celebrated $\Delta I = \frac{1}{2}$ rule of non-leptonic weak decay.

**(b)** For the $\Lambda \to p\pi^-$ and $\Lambda \to n\pi^0$ decays, we have the isospin structure of $\Lambda$ being an isosinglet, and

$$\begin{aligned} |p\pi^-\rangle &= \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \\ |n\pi^0\rangle &= \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle. \end{aligned} \tag{4.209}$$

Thus

$$\begin{aligned} \langle p\pi^-|\mathcal{H}_w|\Lambda\rangle &= \sqrt{\tfrac{1}{3}}\left\langle\tfrac{3}{2}, -\tfrac{1}{2}\right|\mathcal{H}_{3/2}|\Lambda\rangle - \sqrt{\tfrac{2}{3}}\left\langle\tfrac{1}{2}, -\tfrac{1}{2}\right|\mathcal{H}_{1/2}|\Lambda\rangle \\ &= \sqrt{\tfrac{1}{3}}\mathcal{A}_{3/2} - \sqrt{\tfrac{2}{3}}\mathcal{A}_{1/2}, \end{aligned} \tag{4.210}$$

$$\begin{aligned} \left\langle n\pi^0|\mathcal{H}_w|\Lambda\right\rangle &= \sqrt{\tfrac{2}{3}}\left\langle\tfrac{3}{2}, -\tfrac{1}{2}\right|\mathcal{H}_{3/2}|\Lambda\rangle + \sqrt{\tfrac{1}{3}}\left\langle\tfrac{1}{2}, -\tfrac{1}{2}\right|\mathcal{H}_{1/2}|\Lambda\rangle \\ &= \sqrt{\tfrac{2}{3}}\mathcal{A}_{3/2} + \sqrt{\tfrac{1}{3}}\mathcal{A}_{1/2}. \end{aligned} \tag{4.211}$$

where $\mathcal{A}_{3/2} = \left\langle\frac{3}{2}, -\frac{1}{2}\right|\mathcal{H}_{3/2}|\Lambda\rangle$ and $\mathcal{A}_{1/2} = \left\langle\frac{1}{2}, -\frac{1}{2}\right|\mathcal{H}_{1/2}|\Lambda\rangle$. Again the experimental data are in agreement with the $\Delta I = \frac{1}{2}$ rule $\mathcal{A}_{3/2} \ll \mathcal{A}_{1/2}$ prediction of

$$\frac{T(\Lambda \to p\pi^-)}{T(\Lambda \to n\pi^0)} = -\sqrt{2}. \tag{4.212}$$

# 5 Chiral symmetry

## 5.1 Another derivation of Noether's current

Consider a system of scalar fields $\{\phi_i\}$, $i = 1, 2, \ldots, n$, with a Lagrangian depending on $\phi_i$ and $\partial_\mu \phi_i$: $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$. Under an infinitesimal space–time-dependent (local) transformation on $\phi_i$:

$$\phi_i(x) \longrightarrow \phi_i' = \phi_i + \delta\phi_i = \phi_i + \epsilon(x) f_i(\phi) \tag{5.1}$$

where $\epsilon(x)$ is some infinitesimal space–time-dependent parameter and $f_i(\phi)$ is a function of the scalar fields $\{\phi_k\}$.

**(a)** Show that the coefficient of $\partial_\mu \epsilon$ in $\delta\mathcal{L}$ is just the Noether's current as displayed in CL-eqn (5.15):

$$j_\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} f_i. \tag{5.2}$$

**(b)** Show that the coefficient of $\epsilon$ in $\delta\mathcal{L}$ is then the current divergence $\partial^\mu j_\mu(x)$.

### Solution to Problem 5.1

**(a)** Consider the variation of the Lagrangian:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i). \tag{5.3}$$

Because variation and differentiation commute, and because $\delta\phi_i = \epsilon f_i$:

$$\delta(\partial_\mu \phi_i) = \partial_\mu(\delta\phi_i) = (\partial_\mu \epsilon) f_i + \epsilon \partial_\mu f_i \tag{5.4}$$

we have

$$\delta\mathcal{L} = \epsilon \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu f_i \right] + \partial_\mu \epsilon \left[ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} f_i \right]. \tag{5.5}$$

Noether's current (for $\partial_\mu \epsilon = 0$) in CL-eqn (5.15) is just the coefficient of the $\partial_\mu \epsilon$ term in eqn (5.5).

**(b)** The divergence of Noether's current can be evaluated directly:

$$\partial^\mu j_\mu(x) = \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} f_i = \left[ \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\mu f_i. \tag{5.6}$$

Using the Euler–Lagrange equation of motion,

$$\frac{\partial \mathcal{L}}{\partial_\mu \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0, \tag{5.7}$$

we have

$$\partial^\mu j_\mu(x) = \frac{\partial \mathcal{L}}{\partial_\mu \phi_i} f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\mu f_i, \tag{5.8}$$

which is just the coefficient of the $\epsilon$ term of eqn (5.5).

## 5.2 Lagrangian with second derivatives

Consider the case where a Lagrangian depends on $\phi_i$ and $\partial_\mu \phi_i$ as well as the second derivatives:

$$\mathcal{L} = \mathcal{L}(\phi_i,\ \partial_\mu \phi_i,\ \partial_\mu \partial_\nu \phi_i). \tag{5.9}$$

**(a)** Derive the Euler–Lagrange equation of motion for this case.

**(b)** Derive Noether's current for a global transformation:

$$\phi_i(x) \longrightarrow \phi_i' = \phi_i + \delta\phi_i = \phi_i + \epsilon f_i(\phi) \tag{5.10}$$

where $\epsilon$ is some infinitesimal space–time-independent parameter.

**(c)** Show that the current derived in (b) is the same as the coefficient of $\partial_\mu \epsilon$ in $\delta\mathcal{S} = \delta \int \mathcal{L} d^4x$ for a *local* transformation. *Hint:* The higher order derivative terms $\partial_\mu \partial_\nu \epsilon$ can be reduced to $\partial_\mu \epsilon$ upon integration-by-parts.

**Solution to Problem 5.2**

**(a)** The variation of the action

$$\mathcal{S} = \int \mathcal{L}(\phi_i,\ \partial_\mu \phi_i,\ \partial_\mu \partial_\nu \phi_i)\, d^4x$$

being

$$\delta\mathcal{S} = \int \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu(\delta\phi_i) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi_i)} \partial_\mu \partial_\nu(\delta\phi_i) \right] d^4x, \tag{5.11}$$

where we have used the property that differentiation and variation commute,

$$\delta(\partial_\mu \phi_i) = \partial_\mu(\delta\phi_i) \qquad \text{and} \qquad \delta(\partial_\mu \partial_\nu \phi_i) = \partial_\mu \partial_\nu(\delta\phi_i), \tag{5.12}$$

we can obtain the Euler–Lagrange equation of motion through integration-by-parts:

$$\frac{\partial \mathcal{L}}{\partial_\mu \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi_i)} = 0. \tag{5.13}$$

**(b)** For the global transformation, $\delta\phi_i = \epsilon f_i(\phi)$, the variation of the Lagrangian becomes

$$\delta\mathcal{L} = \epsilon \left[ \frac{\partial \mathcal{L}}{\partial \phi_i} f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu f_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi_i)} \partial_\mu \partial_\nu f_i \right]. \tag{5.14}$$

After using the equation of motion, and combining terms with the same number of derivatives:

$$
\begin{aligned}
\delta\mathcal{L} &= \epsilon\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} f_i - \epsilon\partial_\mu\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)} f_i \\
&\quad + \epsilon \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu f_i + \epsilon \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\partial_\mu\partial_\nu f_i \\
&= \epsilon\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} f_i\right] + \epsilon\partial_\mu \left[-\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)} f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\partial_\nu f_i\right].
\end{aligned}
\tag{5.15}
$$

If $\mathcal{L}$ is invariant under the global transformation, the conserved Noether current can then be identified through $\delta\mathcal{L} = \epsilon\partial_\mu j^\mu = 0$:

$$
j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} f_i - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)} f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\partial_\nu f_i.
\tag{5.16}
$$

**Remark.** After applying the equation of motion $\delta\mathcal{L}$ can always be written as the divergence of some 4-vector (which is the essence of Noether's theorem) because the equation of motion follows from $\delta\mathcal{S} = \int \delta\mathcal{L} d^4x = 0$ which comes about because terms having the same number of derivatives combine into a total divergence so that it vanishes upon integration-by-parts.

**(c)** Here we consider the local transformation

$$
\delta\phi_i = \epsilon(x) f_i(\phi) \qquad \text{and} \qquad \partial_\mu(\delta\phi_i) = (\partial_\mu\epsilon) f_i + \epsilon\partial_\mu f_i.
\tag{5.17}
$$

In the variation of the action,

$$
\begin{aligned}
\delta\mathcal{S} &= \int \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\delta\phi_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\partial_\mu\partial_\nu(\delta\phi_i)\right] d^4x \\
&= \int \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\epsilon f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(\partial_\mu\epsilon f_i + \epsilon\partial_\mu f_i) \right.\\
&\qquad \left. -\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}(\partial_\mu\epsilon f_i + \epsilon\partial_\mu f_i)\right] d^4x,
\end{aligned}
\tag{5.18}
$$

the very last term can be rewritten upon integration-by-parts

$$
\begin{aligned}
-\int \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\epsilon\partial_\mu f_i\, d^4x &= -\int \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\epsilon\partial_\nu f_i\, d^4x \\
&= \int \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}(\partial_\mu\epsilon\partial_\nu f_i + \epsilon\partial_\mu\partial_\nu f_i)\, d^4x
\end{aligned}
\tag{5.19}
$$

so that all terms are either proportional to $\epsilon$ or $\partial_\mu\epsilon$:

$$
\begin{aligned}
\delta\mathcal{S} = \int \Bigg\{ &\left[\frac{\partial\mathcal{L}}{\partial\phi_i} f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\partial_\mu\partial_\nu f_i\right]\epsilon(x) \\
&+ \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} f_i - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)} f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi_i)}\partial_\nu f_i\right]\partial_\mu\epsilon \Bigg\} d^4x.
\end{aligned}
\tag{5.20}
$$

From this, we see that the coefficient of $\partial_\mu\epsilon$ is precisely Noether's current.

## 5.3 Conservation laws in a non-relativistic theory

Consider a non-relativistic system described by a Lagrangian $L = L(q_i, \dot{q}_i)$ with $q_i,\ i = 1, 2, \ldots, n$, being the generalized coordinates. Suppose $L$ is invariant under the infinitesimal transformation

$$q_i \longrightarrow q_i' = q_i + i\epsilon t_{ij} q_j. \tag{5.21}$$

**(a)** Show that the quantity (*Noether's charge*) given by

$$Q = \frac{\partial L}{\partial \dot{q}_i} t_{ij} q_j \tag{5.22}$$

is conserved, $dQ/dt = 0$.

**(b)** If the Lagrangian is given as $L = \frac{1}{2} m\mathbf{v}^2 - V(r)$, where $\mathbf{v} = d\mathbf{r}/dt$ with $\mathbf{r} = (x_1, x_2, x_3)$ and $r^2 = x_1^2 + x_2^2 + x_3^2$, show that $L$ is invariant under the infinitesimal rotations

$$x_i \longrightarrow x_i' = x_i + \epsilon_{ij} x_j \tag{5.23}$$

where $\epsilon_{ij} = -\epsilon_{ji}$. Explicitly construct the conserved charges.

**(c)** For the case where $L = \frac{1}{2} m\mathbf{v}^2$, show that $L$ is invariant under the spatial translations

$$\mathbf{r} \longrightarrow \mathbf{r}' = \mathbf{r} + \mathbf{a} \tag{5.24}$$

where $\mathbf{a}$ is an arbitrary constant vector. Find the conserved charges.

**(d)** Consider a system of two particles interacting with each other through a potential which depends only on the relative coordinates $V(\mathbf{r}_1 - \mathbf{r}_2)$. Show that the total momenta $\mathbf{p} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ are conserved.

**Solution to Problem 5.3**

**(a)** The variation of the Lagrangian is given by

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i. \tag{5.25}$$

Using $\delta\dot{q}_i = (d/dt)(\delta q_i)$ and the equation of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \tag{5.26}$$

we get

$$\begin{aligned}\delta L &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = i\epsilon\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} t_{ij} q_j\right).\end{aligned} \tag{5.27}$$

Thus the invariance of the Lagrangian $\delta L = 0$ implies the charge conservation $dQ/dt = 0$, with the charge being $Q = (\partial L/\partial \dot{q}_i) t_{ij} q_j$.

**(b)** We can see that

$$\delta L = \frac{1}{2} m \delta \mathbf{v}^2 - \frac{\partial V(r)}{\partial r} \delta r = 0 \tag{5.28}$$

follows from $\epsilon_{ij} = -\epsilon_{ji}$ because

$$\delta r = \frac{\partial r}{\partial x_i} \delta x_i = \frac{x_i}{r} \delta x_i = \frac{1}{r} x_i \epsilon_{ij} x_j = 0 \tag{5.29}$$

and

$$\delta \mathbf{v}^2 = 2\dot{x}_i \delta \dot{x}_i = 2\dot{x}_i \epsilon_{ij} \dot{x}_i = 0. \tag{5.30}$$

We can use the expression in eqn (5.27)

$$\delta L = \epsilon_{ij} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} x_j \right) = \tfrac{1}{2} \epsilon_{ij} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} x_j - \frac{\partial L}{\partial \dot{x}_j} x_i \right) \tag{5.31}$$

to find the conserved charge for this rotational symmetry,

$$Q_{ij} = \frac{\partial L}{\partial \dot{x}_i} x_j - \frac{\partial L}{\partial \dot{x}_j} x_i = m\dot{x}_i x_j - m\dot{x}_j x_i = p_i x_j - p_j x_i \tag{5.32}$$

which are just the familiar angular momenta.

**(c)** Because $\mathbf{a}$ is a constant, we have $\mathbf{v}' = \mathbf{v}$ and thus $\delta L = \frac{1}{2} m \delta \mathbf{v}^2 = 0$. To obtain the charge of this spatial translational symmetry, we note that, for an infinitesimally small $\mathbf{a}$, one has $\delta x_i = a_i$ and thus

$$\delta L = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \delta x_i \right) = a_i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) = a_i \frac{d}{dt} (m\dot{x}_i). \tag{5.33}$$

Consequently $\delta L = 0$ leads to the conserved charges $m\dot{x}_i$, which are just the familiar linear momenta.

**(d)** The Lagrangian for this case is given by

$$L = \frac{1}{2} m_1 \mathbf{v}_1^2 + \frac{1}{2} m_2 \mathbf{v}_2^2 - V(\mathbf{r}_1 - \mathbf{r}_2). \tag{5.34}$$

Clearly, $L$ is invariant under the spatial translation of the form

$$\mathbf{r}_1 \longrightarrow \mathbf{r}_1' = \mathbf{r}_1 + \mathbf{a}, \qquad \mathbf{r}_2 \longrightarrow \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{a}. \tag{5.35}$$

For infinitesimal translations, we have $\delta \mathbf{r}_1 = \delta \mathbf{r}_2 = \mathbf{a}$ and

$$\begin{aligned} \delta L &= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}_{1i}} \delta r_{1i} + \frac{\partial L}{\partial \dot{r}_{2i}} \delta r_{2i} \right) \\ &= a_i \frac{d}{dt} (m_1 \dot{r}_{1i} + m_2 \dot{r}_{2i}). \end{aligned} \tag{5.36}$$

Thus the total momentum

$$\mathbf{p} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 \tag{5.37}$$

is conserved.

## 5.4 Symmetries of the linear $\sigma$-model

The Lagrangian for the linear $\sigma$-model is given by

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma)^2\right] + \bar{N} i \gamma^\mu \partial_\mu N \\ &\quad + g\bar{N}(\sigma + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi})N + \frac{\mu^2}{2}\left(\sigma^2 + \boldsymbol{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2\right)^2 \end{aligned} \tag{5.38}$$

where $N = \binom{p}{n}$ is an isospin-$\frac{1}{2}$ nucleon field, $\boldsymbol{\pi}=(\pi_1, \pi_2, \pi_3)$ an isospin one pion field, and $\sigma$ an isospin zero scalar field. It is convenient to use a $2 \times 2$ matrix to represent the spin 0 fields collectively:

$$\Sigma = \sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}. \tag{5.39}$$

**(a)** Show that the Lagrangian is invariant under the isospin transformations:

$$N \longrightarrow N' = UN, \qquad \Sigma \longrightarrow \Sigma' = U\Sigma U^\dagger, \tag{5.40}$$

where $U = \exp\left(\frac{i}{2}\boldsymbol{\alpha} \cdot \boldsymbol{\tau}\right)$ is an arbitrary $2 \times 2$ unitary matrix with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ being a set of real constants. Find the corresponding conserved isospin vector currents $V_\mu^i,\ i = 1, 2, 3$.

**(b)** Show that the Lagrangian is invariant under the axial isospin transformations

$$N \longrightarrow N' = \exp\left(i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma_5\right)N, \qquad \Sigma \longrightarrow \Sigma' = V^\dagger \Sigma V^\dagger, \tag{5.41}$$

where $V = \exp\left(\frac{i}{2}\boldsymbol{\beta} \cdot \boldsymbol{\tau}\right)$ is an arbitrary $2 \times 2$ unitary matrix with $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ being a set of real constants. Find the corresponding conserved axial-vector currents $A_\mu^i$.

**(c)** Calculate the charge commutators

$$[Q^i, Q^j], \qquad [Q^i, Q^{5j}], \qquad [Q^{5i}, Q^{5j}], \tag{5.42}$$

where

$$Q^i = \int d^3x\, V_0^i(x) \qquad \text{and} \qquad Q^{5i} = \int d^3x\, A_0^i(x).$$

**(d)** Calculate the commutators of particle fields with the vector charges:

$$[Q^i, N^a], \qquad [Q^i, \pi^j], \qquad [Q^i, \sigma] \tag{5.43}$$

and with the axial-vector charges:

$$[Q^{5i}, N^a], \qquad [Q^{5i}, \pi^j], \qquad [Q^{5i}, \sigma]. \tag{5.44}$$

**Solution to Problem 5.4**

**(a)** It is useful to define the left-handed and right-handed chiral nucleon fields:

$$N_L = \frac{1}{2}(1-\gamma_5)N \qquad \text{and} \qquad N_R = \frac{1}{2}(1+\gamma_5)N. \tag{5.45}$$

Thus $N = N_L + N_R$, $\gamma_5 N_L = -N_L$, and $\gamma_5 N_R = +N_R$. Also,

$$\Sigma\Sigma^\dagger = (\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\sigma - i\boldsymbol{\tau}\cdot\boldsymbol{\pi}) = \sigma^2 + (\boldsymbol{\tau}\cdot\boldsymbol{\pi})^2 = (\sigma^2+\boldsymbol{\pi}^2)\mathbf{I}, \tag{5.46}$$

where we have used the Pauli matrix identity

$$(\boldsymbol{\tau}\cdot\mathbf{A})(\boldsymbol{\tau}\cdot\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B}) + i\boldsymbol{\tau}\cdot(\mathbf{A}\times\mathbf{B}). \tag{5.47}$$

We can then write the Lagrangian as

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}tr\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right) + \bar{N}_L i\gamma^\mu\partial_\mu N_L + \bar{N}_R i\gamma^\mu\partial_\mu N_R \\ &\quad + g\left(\bar{N}_L\Sigma N_R + \bar{N}_R\Sigma^\dagger N_L\right) + \frac{\mu^2}{4}tr(\Sigma\Sigma^\dagger) - \frac{\lambda}{16}[tr(\Sigma\Sigma^\dagger)]^2. \end{aligned} \tag{5.48}$$

For the isospin rotations, we have

$$N_L \longrightarrow N'_L = UN_L, \quad N_R \longrightarrow N'_R = UN_R, \quad \text{and} \quad \Sigma \longrightarrow \Sigma' = U\Sigma U^\dagger. \tag{5.49}$$

Thus,

$$tr(\Sigma'\Sigma'^\dagger) = tr((U\Sigma U^\dagger)(U\Sigma^\dagger U^\dagger)) = tr(\Sigma\Sigma^\dagger) \tag{5.50}$$

and, in the same way, $tr(\partial_\mu\Sigma'\partial^\mu\Sigma'^\dagger) = tr(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)$. Also,

$$\bar{N}'_L i\gamma^\mu\partial_\mu N'_L = \bar{N}_L U^\dagger i\gamma^\mu\partial_\mu U N_L = \bar{N}_L i\gamma^\mu\partial_\mu N_L \tag{5.51}$$

and, similarly, for the $\bar{N}_R i\gamma^\mu\partial_\mu N_R$ term. Furthermore,

$$\bar{N}'_L\Sigma' N'_R = \bar{N}_L U^\dagger U\Sigma U^\dagger U N_R = \bar{N}_L\Sigma N_R. \tag{5.52}$$

Thus $\mathcal{L}$ is invariant under the isospin rotation. To get the conserved current, we need to work out the infinitesimal transformations:

$$N \longrightarrow N' = \exp\left(i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right)N \simeq \left(1 + i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right)N \tag{5.53}$$

or

$$\delta N = i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}N. \tag{5.54}$$

Also,

$$\begin{aligned}\Sigma \longrightarrow \Sigma' &\simeq \left(1 + i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right)(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\left(1 - i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right) \\ &\simeq \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} + i^2\left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2},\ \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right]. \end{aligned} \tag{5.55}$$

From $[\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k$ we can work out the last commutator to be

$$\left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2},\ \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right] = i\epsilon_{ijk}\alpha_i\pi_j\tau_k = i(\boldsymbol{\alpha}\times\boldsymbol{\pi})\cdot\boldsymbol{\tau}. \tag{5.56}$$

Thus,

$$\Sigma' = \sigma' + i\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}' \simeq \sigma + i\,\boldsymbol{\tau}\cdot\boldsymbol{\pi} - i\,\boldsymbol{\tau}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\pi}) \tag{5.57}$$

or

$$\sigma' = \sigma, \quad \boldsymbol{\pi}' = \boldsymbol{\pi} - \boldsymbol{\alpha}\times\boldsymbol{\pi}. \tag{5.58}$$

Namely,

$$\delta\sigma = 0, \quad \delta\boldsymbol{\pi} = -\boldsymbol{\alpha}\times\boldsymbol{\pi}. \tag{5.59}$$

The conserved isospin vector current $\mathbf{V}_\mu$ is simply the Noether current for this symmetry transformation:

$$\begin{aligned} -\boldsymbol{\alpha}\cdot\mathbf{V}^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu N)}\delta N + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\sigma)}\delta\sigma + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\boldsymbol{\pi})}\delta\boldsymbol{\pi} \\ &= -\bar{N}\gamma^\mu\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}N - \partial^\mu\boldsymbol{\pi}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\pi}). \end{aligned} \tag{5.60}$$

Or

$$\mathbf{V}^\mu = \bar{N}\gamma^\mu\frac{\boldsymbol{\tau}}{2}N - \partial^\mu\boldsymbol{\pi}\times\boldsymbol{\pi}. \tag{5.61}$$

**(b)** The axial transformation of the nucleon field

$$N \longrightarrow N' = \exp\left(i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma_5\right)N \tag{5.62}$$

takes on a simple form when expressed in terms of its chiral components

$$N_L' = V^\dagger N_L \qquad \text{and} \qquad N_R' = VN_R \tag{5.63}$$

where $V = \exp\left(i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\right)$. It is then easy to see that both $\bar{N}_L i\gamma^\mu\partial_\mu N_L$ and $\bar{N}_R i\gamma^\mu\partial_\mu N_R$ are invariants. Similarly, both

$$tr(\Sigma'\Sigma'^\dagger) = tr((V^\dagger\Sigma V^\dagger)(V\Sigma^\dagger V)) = tr(\Sigma\Sigma^\dagger) \tag{5.64}$$

and $tr(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)$ are also invariants. For the Yukawa couplings,

$$\bar{N}_L'\Sigma' N_R' = \bar{N}_L V(V^\dagger\Sigma V^\dagger)VN_R = \bar{N}_L\Sigma N_R. \tag{5.65}$$

Hence $\mathcal{L}$ is invariant under the axial isospin rotations. To get the conserved current, we need to work out the infinitesimal transformations

$$N \longrightarrow N' = \exp\left(i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma_5\right)N \simeq \left(1 + i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma_5\right)N \tag{5.66}$$

or the infinitesimally small change of the nucleon field under axial transformation being

$$\delta_5 N = i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma_5 N \tag{5.67}$$

and

$$\begin{aligned}\Sigma \longrightarrow \Sigma' &\simeq \left(1 - i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\right)(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\left(1 - i\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\right)\\ &\simeq \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} - i(\boldsymbol{\beta}\cdot\boldsymbol{\tau})\sigma + \left\{\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2},\ \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right\} + \cdots .\end{aligned} \tag{5.68}$$

From $\{\tau_i,\ \tau_j\} = 2\delta_{ij}$ we can work out the last commutator to be

$$\left\{\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2},\ \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right\} = \boldsymbol{\beta}\cdot\boldsymbol{\pi}. \tag{5.69}$$

Thus,

$$\Sigma' = \sigma' + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}' \simeq \sigma + (\boldsymbol{\beta}\cdot\boldsymbol{\pi}) + i(\boldsymbol{\tau}\cdot\boldsymbol{\pi}) - i\sigma(\boldsymbol{\beta}\cdot\boldsymbol{\tau}) \tag{5.70}$$

or

$$\sigma' = \sigma + \boldsymbol{\beta}\cdot\boldsymbol{\pi}, \quad \boldsymbol{\pi}' = \boldsymbol{\pi} - \sigma\boldsymbol{\beta}. \tag{5.71}$$

Namely,

$$\delta_5\sigma = \boldsymbol{\beta}\cdot\boldsymbol{\pi}, \quad \delta_5\boldsymbol{\pi} = -\sigma\boldsymbol{\beta}. \tag{5.72}$$

The conserved isospin axial-vector current $\mathbf{A}_\mu$ is simply the Noether current for this symmetry transformation:

$$\begin{aligned}-\boldsymbol{\beta}\cdot\mathbf{A}^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu N)}\delta_5 N + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\sigma)}\delta_5\sigma + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\boldsymbol{\pi})}\delta_5\boldsymbol{\pi}\\ &= -\bar{N}\gamma^\mu\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma_5 N + \partial^\mu\sigma(\boldsymbol{\beta}\cdot\boldsymbol{\pi}) - \sigma\partial^\mu\boldsymbol{\pi}\cdot\boldsymbol{\beta}.\end{aligned} \tag{5.73}$$

Or

$$\mathbf{A}^\mu = \bar{N}\gamma^\mu\gamma_5\frac{\boldsymbol{\tau}}{2}N - (\boldsymbol{\pi}\partial^\mu\sigma - \sigma\partial^\mu\boldsymbol{\pi}). \tag{5.74}$$

**Remark.** We can combine the transformations in (a) and (b) as follows:

$$N_R \longrightarrow N_R' = RN_R \tag{5.75}$$

$$N_L \longrightarrow N_L' = LN_L \tag{5.76}$$

$$\Sigma \longrightarrow \Sigma' = L\Sigma R^\dagger \tag{5.77}$$

where we have introduced the right-handed and left-hand transformations

$$R = \exp\left(i\frac{\boldsymbol{\gamma}\cdot\boldsymbol{\tau}}{2}\right) \qquad \text{and} \qquad L = \exp\left(i\frac{\boldsymbol{\delta}\cdot\boldsymbol{\tau}}{2}\right) \tag{5.78}$$

with $\boldsymbol{\gamma} = \boldsymbol{\delta} = \boldsymbol{\alpha}$ for the vector transformation, and $\boldsymbol{\gamma} = -\boldsymbol{\delta} = \boldsymbol{\beta}$ for the axial transformation. Under the infinitesimal transformation, $R \simeq (1 + i\frac{\gamma\cdot\tau}{2})$:

$$\delta_R N_R = i\frac{\boldsymbol{\gamma}\cdot\boldsymbol{\tau}}{2}N_R \qquad \text{and} \qquad \delta_R N_L = 0 \tag{5.79}$$

and

$$\begin{aligned}\Sigma' &\simeq (\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\left(1 - i\frac{\boldsymbol{\gamma}\cdot\boldsymbol{\tau}}{2}\right)\\ &\simeq \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} - i\frac{\boldsymbol{\gamma}\cdot\boldsymbol{\tau}}{2}\sigma + \frac{\boldsymbol{\gamma}\cdot\boldsymbol{\pi}}{2} + i\frac{\boldsymbol{\pi}\times\boldsymbol{\gamma}}{2}\cdot\tau\end{aligned} \tag{5.80}$$

or

$$\delta_R\sigma = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{\pi}}{2}, \qquad \delta_R\boldsymbol{\pi} = \frac{\boldsymbol{\pi}\times\boldsymbol{\gamma}}{2} - \frac{\boldsymbol{\gamma}}{2}\sigma. \tag{5.81}$$

From these field variations, we can immediately work out the corresponding conserved current

$$\mathbf{R}^\mu = \bar{N}_R\gamma^\mu\frac{\boldsymbol{\tau}}{2}N_R - \frac{\boldsymbol{\pi}}{2}\partial^\mu\sigma - \frac{1}{2}(\partial^\mu\boldsymbol{\pi}\times\boldsymbol{\pi} - \sigma\partial^\mu\boldsymbol{\pi}). \tag{5.82}$$

Similarly for the left-handed transformation, $L \simeq (1 + \frac{i}{2}\boldsymbol{\delta}\cdot\boldsymbol{\tau})$:

$$\delta_L N_R = 0 \qquad \text{and} \qquad \delta_L N_L = i\frac{\boldsymbol{\delta}\cdot\boldsymbol{\tau}}{2}N_L \tag{5.83}$$

and

$$\begin{aligned}\Sigma' &\simeq \left(1 + i\frac{\boldsymbol{\delta}\cdot\boldsymbol{\tau}}{2}\right)(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\\ &\simeq \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} + i\frac{\boldsymbol{\delta}\cdot\boldsymbol{\tau}}{2}\sigma - \frac{\boldsymbol{\delta}\cdot\boldsymbol{\pi}}{2} + i\frac{\boldsymbol{\pi}\times\boldsymbol{\delta}}{2}\cdot\boldsymbol{\tau}\end{aligned} \tag{5.84}$$

or

$$\delta_L\sigma = -\frac{\boldsymbol{\delta}\cdot\boldsymbol{\pi}}{2}, \qquad \delta_L\boldsymbol{\pi} = \frac{\boldsymbol{\pi}\times\boldsymbol{\delta}}{2} + \frac{\boldsymbol{\delta}}{2}\sigma, \tag{5.85}$$

leading to the conserved current

$$\mathbf{L}^\mu = \bar{N}_L\gamma^\mu\frac{\boldsymbol{\tau}}{2}N_L + \frac{\boldsymbol{\pi}}{2}\partial^\mu\sigma - \frac{1}{2}(\partial^\mu\boldsymbol{\pi}\times\boldsymbol{\pi} + \sigma\partial^\mu\boldsymbol{\pi}). \tag{5.86}$$

The vector and axial-vector currents are then given as

$$\mathbf{V}^\mu = \mathbf{R}^\mu + \mathbf{L}^\mu \qquad \text{and} \qquad \mathbf{A}^\mu = \mathbf{R}^\mu - \mathbf{L}^\mu. \tag{5.87}$$

**(c)** Let us first consider the vector charges

$$Q^i = \int d^3x V_0^i(x). \tag{5.88}$$

Because of current conservation, $\partial^\mu V_\mu^i = 0$, the charges $Q^i$ must be time-independent. We can then choose to work with equal-time commutators:

$$[Q^i, Q^j]_{x_0=y_0} = \int d^3x d^3y \left[N^\dagger \frac{\tau_i}{2} N - \epsilon_{ilm}\partial^0\pi^l\pi^m, N^\dagger \frac{\tau_j}{2} N - \epsilon_{jnk}\partial^0\pi^n\pi^k\right] \tag{5.89}$$

which can be evaluated, see Problem 4.6, as

$$\begin{aligned}\left[N^\dagger(x)\frac{\tau_i}{2}N(x), N^\dagger(y)\frac{\tau_j}{2}N(y)\right]_{x_0=y_0} &= N^\dagger(x)\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right]N(y)\delta^3(\mathbf{x}-\mathbf{y}) \\ &= i\epsilon_{ijk}\left(N^\dagger\frac{\tau_k}{2}N\right)\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned}\tag{5.90}$$

and

$$\begin{aligned}&\epsilon_{ilm}\epsilon_{jnk}[\partial^0\pi^l(x)\pi^m(x),\ \partial^0\pi^n(y)\pi^k(y)]_{x_0=y_0} \\ &\quad = \epsilon_{ilm}\epsilon_{jnk}\left(-i\delta^{lk}\,\partial^0\pi^n(y)\pi^m(x) + i\delta^{mn}\partial^0\pi^l(x)\pi^k(y)\right)\delta^3(\mathbf{x}-\mathbf{y}) \\ &\quad = -i\left(\epsilon_{ikm}\epsilon_{jnk}\,\partial^0\pi^n\pi^m - \epsilon_{ilm}\epsilon_{jmk}\partial^0\pi^l\pi^k\right)\delta^3(\mathbf{x}-\mathbf{y}) \\ &\quad = -i(\partial^0\pi^i\pi^j - \delta^{ij}\partial^0\pi^n\pi^n + \delta^{ij}\partial^0\pi^l\pi^l - \partial^0\pi^j\pi^i)\delta^3(\mathbf{x}-\mathbf{y}) \\ &\quad = -i\epsilon_{ijk}\epsilon_{klm}(\partial^0\pi^l)\pi^m\delta^3(\mathbf{x}-\mathbf{y}). \end{aligned}\tag{5.91}$$

After substituting the results of eqns (5.90) and (5.91) into (5.89):

$$[Q^i, Q^j]_{x_0=y_0} = i\epsilon_{ijk}\int d^3x\left(N^\dagger\frac{\tau_k}{2}N - \epsilon_{klm}\partial^0\pi^l\pi^m\right) = i\epsilon_{ijk}Q^k. \tag{5.92}$$

In a similar manner, we can verify the other commutation relations of

$$[Q^i, Q^{5j}] = i\epsilon_{ijk}Q^{5k} \qquad \text{and} \qquad [Q^{5i}, Q^{5j}] = i\epsilon_{ijk}Q^k. \tag{5.93}$$

**Remark.** If we define the right-handed and left-handed charges as

$$Q_R^i = Q^i + Q^{5j}, \qquad Q_L^i = Q^i - Q^{5j}, \tag{5.94}$$

the algebra becomes

$$\left[Q_L^i, Q_L^j\right] = i\epsilon_{ijk}Q_L^k, \qquad \left[Q_R^i, Q_R^j\right] = i\epsilon_{ijk}Q_R^k \tag{5.95}$$

and

$$\left[Q_L^i, Q_R^j\right] = 0. \tag{5.96}$$

Namely, each set of $\{Q_{\mathrm{L}}^i\}$ and $\{Q_{\mathrm{R}}^i\}$ separately form an SU(2) algebra. This is why it is referred to as the $SU(2)_L \times SU(2)_R$ algebra.

**(d)** Here we calculate the commutator of the various fields with the isospin charge

$$Q^i = \int d^3x \left( N^\dagger(x) \frac{\tau^i}{2} N(x) - \epsilon^{ijk} \partial^0 \pi^j(x) \pi^k(x) \right). \tag{5.97}$$

For the isodoublet nucleon field,

$$[Q^i, N^a(y)] = \int d^3x \left[ N^{b\dagger}(x) \left( \frac{\tau^i}{2} \right)^{bc} N^c(x), N^a(y) \right], \tag{5.98}$$

we can use the identity

$$[AB, C] = A\{B, C\} - \{A, C\}B \tag{5.99}$$

to get

$$[Q^i, N^a(y)] = -\left( \frac{\tau^i}{2} \right)^{ac} N^c(y). \tag{5.100}$$

For the isotriplet pion field,

$$\begin{aligned} [Q^i, \pi^l(y)] &= -\epsilon^{ijk} \int d^3x [\partial^0 \pi^j(x) \pi^k(x), \pi^l(y)] \\ &= -\epsilon^{ijk} \int d^3x \big( -i\delta^{jl} \big) \pi^k(x) \delta^3(\mathbf{x} - \mathbf{y}) \qquad (5.101) \\ &= i\epsilon^{ilk} \pi^k(y). \qquad (5.102) \end{aligned}$$

For the singlet $\sigma$-field,

$$[Q^i, \sigma(y)] = 0. \tag{5.103}$$

By comparing these results with the eqns (5.54) and (5.58) obtained in Part (a), we note that these charge-field commutators just yield the variation of the field under the isospin transformation with some parameter $\alpha^i$:

$$[\boldsymbol{\alpha} \cdot \mathbf{Q}, \phi(x)] = i\delta\phi(x), \tag{5.104}$$

where $\phi(x) = N(x), \pi^i(x)$, or $\sigma(x)$. Similarly, for the axial charge, we have $[\boldsymbol{\beta} \cdot \mathbf{Q}, \phi(x)] = i\delta_5\phi(x)$:

$$[Q^{5i}, N^a(y)] = -\left( \frac{\tau^i}{2} \right)^{ac} \gamma_5 N^c(y), \tag{5.105}$$

$$[Q^{5i}, \pi^l(y)] = -i\delta^{ij}\sigma(y), \tag{5.106}$$

$$[Q^{5i}, \sigma(y)] = i\pi^i(y). \tag{5.107}$$

## 5.5 Spontaneous symmetry breaking in the $\sigma$-model

In Problem 5.4 the effective potential for the scalar fields has the form of

$$V = -\frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2. \tag{5.108}$$

For the case of $\mu^2 > 0$, the minimum of this potential is at [see CL-eqn (5.168)],

$$\sigma^2 + \boldsymbol{\pi}^2 = v^2, \qquad v = \left(\frac{\mu^2}{\lambda}\right)^{1/2}. \tag{5.109}$$

In the text, we chose the vacuum configuration to be

$$\langle \pi^1 \rangle = \langle \pi^2 \rangle = \langle \pi^3 \rangle = 0, \quad \langle \sigma \rangle = v. \tag{5.110}$$

Now consider the alternative configuration of

$$\langle \pi^1 \rangle = \langle \pi^2 \rangle = \langle \sigma \rangle = 0, \quad \langle \pi^3 \rangle = v. \tag{5.111}$$

**(a)** Show that the charges which do not annihilate the vacuum are $Q^1$, $Q^2$, and $Q^{53}$, and the Goldstone bosons are $\pi^1$, $\pi^2$, and $\sigma$ fields.

**(b)** Show that the remaining charges, $Q^{51}$, $Q^{52}$, and $Q^3$, form an SU(2) algebra.

**(c)** Show that the fermion bilinear $\mathcal{L}_m = gv\bar{N}i\gamma_5\tau^3 N$ generated by $\langle \pi^3 \rangle = v$ can be transformed into the standard fermion mass term of $\mathcal{L}_m = m_N \bar{N}'N'$ by some chiral rotation. Find this transformation.

**Solution to Problem 5.5**

**(a)** Given that $\langle \pi^3 \rangle \neq 0$, we seek charge-field commutators eqns (5.100)–(5.107) which are proportional to the $\pi^3$ field:

$$[Q^1, \pi^2] = -[Q^2, \pi^1] = [Q^{53}, \sigma] = i\pi^3. \tag{5.112}$$

We see that the charges $Q^1$, $Q^2$, and $Q^{53}$ do not annihilate the vacuum [otherwise the above equation would imply that $\langle \pi^3 \rangle = 0$], and $\pi^1$, $\pi^2$, and $\sigma$ are Goldstone boson fields.

**(b)** From the charge commutators calculated in Problem 4 we have

$$[Q^{51}, Q^{52}] = iQ^3, \quad -[Q^3, Q^{51}] = iQ^{52}, \quad [Q^{52}, Q^3] = iQ^{51}. \tag{5.113}$$

This means that the charges $Q^{51}$, $Q^{52}$, and $Q^3$ form an SU(2) algebra.

**(c)** To find the chiral rotation it is useful to decompose the fermion field into its chiral components, $N = N_L + N_R$. In this way we have

$$\mathcal{L}_m = gv\bar{N}i\gamma_5\tau^3 N = gv\left[\bar{N}_L i\tau^3 N_R - \bar{N}_R i\tau^3 N_L\right]. \tag{5.114}$$

Consider the chiral transformation

$$N_L = LN_L', \qquad N_R = RN_R', \tag{5.115}$$

where $L$ and $R$ are unitary transformations. To turn

$$\mathcal{L}_m = gv\left[\bar{N}'_L L^\dagger i\tau^3 R N'_R - \bar{N}'_R R^\dagger i\tau^3 L N'_L\right] \tag{5.116}$$

into the standard fermion mass term of $-\left(\bar{N}'_L N'_R + \bar{N}'_R N'_L\right)$, we require

$$L^\dagger i\tau^3 R = -1, \qquad R^\dagger i\tau^3 L = 1, \tag{5.117}$$

which is actually one condition as $(L^\dagger i\tau^3 R)^\dagger = -R^\dagger i\tau^3 L = -1$. One simple solution to this condition, hence the required chiral rotation, is

$$L = -i\tau^3 = \exp\left(-i\pi\frac{\tau^3}{2}\right), \qquad R = 1, \tag{5.118}$$

where we have used the identity of

$$\exp\left(i\theta\frac{\tau^3}{2}\right) = \cos\frac{\theta}{2} + i\tau^3\sin\frac{\theta}{2}. \tag{5.119}$$

**Remark.** In spontaneous symmetry breaking, the choice of the vacuum expectation value (VEV) direction is a matter of convention. All different choices yielding the same symmetry-breaking pattern are physically equivalent. In the example under discussion, both choices of $\langle\pi^3\rangle \neq 0$ and $\langle\sigma\rangle \neq 0$ give the same symmetry breaking, SU(2) × SU(2) → SU(2) or equivalently SO(4) → SO(3), and have exactly the same physical content.

## 5.6 PCAC in the $\sigma$-model

Suppose we introduce a symmetry-breaking term into the $\sigma$-model Lagrangian

$$\mathcal{L}_{SB} = -c\sigma(x) \tag{5.120}$$

where $c$ is a constant.

**(a)** Find the new minimum for the effective potential,

$$V = -\frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2 + c\sigma. \tag{5.121}$$

**(b)** Show that in this case, pions are no longer massless and, in the tree level, their masses are proportional to the constant $c$.

**(c)** Show that the axial-vector current $\mathbf{A}_\mu$ derived in Problem 5.4 is no longer conserved. Calculate the divergence $\partial^\mu \mathbf{A}_\mu$ and show that it is proportional to the pion field $\boldsymbol{\pi}$.

**Solution to Problem 5.6**

**(a)** We have the minimization conditions

$$\frac{\partial V}{\partial \sigma} = [-\mu^2 + \lambda(\sigma^2 + \boldsymbol{\pi}^2)]\sigma + c = 0 \tag{5.122}$$

and

$$\frac{\partial V}{\partial \pi^i} = [-\mu^2 + \lambda(\sigma^2 + \boldsymbol{\pi}^2)]\pi^i = 0. \tag{5.123}$$

Since it is not possible to have $[-\mu^2 + \lambda(\sigma^2 + \boldsymbol{\pi}^2)] = 0$ for a non-vanishing $c$, we must have $\pi^i = 0$ and the $\sigma$-field satisfying the cubic equation:

$$-\mu^2\sigma + \lambda\sigma^3 + c = 0. \tag{5.124}$$

**Remark.** In this case the vacuum configuration is unique because the symmetry-breaking term of eqn (5.120) has singled out a direction.

**(b)** To discover the physical content of the model, we shift the fields

$$\boldsymbol{\pi}' = \boldsymbol{\pi}, \qquad \sigma' = \sigma - v, \tag{5.125}$$

where $v$ is the solution to the cubic equation $-\mu^2 v + \lambda v^3 + c = 0$. The terms in the effective potential become

$$\begin{aligned} \sigma^2 + \boldsymbol{\pi}^2 &= \sigma'^2 + \boldsymbol{\pi}'^2 + 2v\sigma' + v^2 \\ (\sigma^2 + \boldsymbol{\pi}^2)^2 &= 4v^2\sigma'^2 + 2v^2(\sigma'^2 + \boldsymbol{\pi}'^2) + \text{ non-quadratic terms.} \end{aligned} \tag{5.126}$$

We then have the mass terms in the effective potential

$$\begin{aligned} V_2 &= -\frac{\mu^2}{2}\left(\sigma'^2 + \boldsymbol{\pi}'^2\right) + \frac{\lambda}{4}\left[4v^2\sigma'^2 + 2v^2(\sigma'^2 + \boldsymbol{\pi}'^2)\right] \\ &= \left(\frac{3\lambda}{2}v^2 - \frac{\mu^2}{2}\right)\sigma'^2 + \left(\frac{\lambda}{2}v^2 - \frac{\mu^2}{2}\right)\boldsymbol{\pi}'^2 \\ &= \left(\mu^2 - \frac{3c}{2v}\right)\sigma'^2 - \frac{c}{2v}\boldsymbol{\pi}'^2 \end{aligned} \tag{5.127}$$

where to reach the last line we have used the cubic equation for $v$. Thus

$$m_\sigma^2 = 2\mu^2 - \frac{3c}{v} \tag{5.128}$$

$$m_\pi^2 = -\frac{c}{v}. \tag{5.129}$$

If we pick $c < 0$, then both $m_\sigma^2$ and $m_\pi^2$ are positive. That all three components of $\boldsymbol{\pi}$ have equal mass means that the explicit breaking $\mathcal{L}_{SB} = -c\sigma(x)$ still leaves isospin SU(2) symmetry unbroken.

**(c)** The divergence of the axial current is related to the variation of the Lagrangian as [see CL-eqn (5.14) or Problem 5.1]

$$\boldsymbol{\beta} \cdot \partial^\mu \mathbf{A}_\mu = \delta_5 \mathcal{L}. \tag{5.130}$$

The right-hand side would vanish were it not for the presence of the symmetry-breaking term $\mathcal{L}_{SB} = -c\sigma(x)$. Thus

$$\delta_5 \mathcal{L} = \delta_5 \mathcal{L}_{SB} = \frac{\partial \mathcal{L}_{SB}}{\partial(\sigma)} \delta_5 \sigma = -c \boldsymbol{\beta} \cdot \boldsymbol{\pi}, \tag{5.131}$$

where we have used a result obtained in Problem 5.4, eqn (5.72). In this way we find

$$\partial^\mu \mathbf{A}_\mu = -c\boldsymbol{\pi}. \tag{5.132}$$

**Remark 1.** The constant $c$ can be related to the pion mass and the pion decay constant, $m_\pi$ and $f_\pi$. For the $\pi \to \mu\nu_\mu$ decay, the amplitude is proportional to the axial current matrix element, which defines $f_\pi$ by

$$\langle 0|A^a_\mu(0)|\pi^b(p)\rangle = i\delta^{ab} f_\pi p_\mu. \tag{5.133}$$

Thus the matrix element of the divergence is given by

$$\langle 0|\partial^\mu A^a_\mu(0)|\pi^b(p)\rangle = \delta^{ab} f_\pi m_\pi^2 = -c\langle 0|\pi^a(0)|\pi^b(p)\rangle. \tag{5.134}$$

Or

$$-c = f_\pi m_\pi^2. \tag{5.135}$$

In this way, the divergence has the PCAC form

$$\partial^\mu A^a_\mu = f_\pi m_\pi^2 \pi^a. \tag{5.136}$$

The specific value of the pion decay constant is fixed as follows. The amplitude for the $\pi^+ \to \mu^+\nu_\mu$ decay can be written as

$$T = \frac{G_F}{\sqrt{2}} \langle 0|A^-_\mu(0)|\pi^+(p)\rangle \bar{\mu}\gamma^\mu(1-\gamma_5)v \tag{5.137}$$

with

$$\langle 0|A^-_\mu(0)|\pi^+(p)\rangle = i\sqrt{2} f_\pi p_\mu. \tag{5.138}$$

With this definition, one finds from the decay rate that [see Problem 11.3(c) for the calculation]

$$f_\pi = \frac{0.93}{\sqrt{2}} m_\pi = 92\,\text{MeV}. \tag{5.139}$$

**Remark 2.** Comparing eqns (5.129) and (5.135) we see that the VEV of the σ-field is simply the pion decay constant:

$$v = f_\pi. \tag{5.140}$$

## 5.7 Non-linear $\sigma$-model I

In the $\sigma$-model of Problem 5.4, the combination $\Sigma\Sigma^\dagger = \sigma^2 + \boldsymbol{\pi}^2$ is invariant under the SU(2) × SU(2) transformation. The non-linear $\sigma$-model is obtained from the linear $\sigma$-model by imposing the constraint of

$$\sigma^2 + \boldsymbol{\pi}^2 = f^2 \qquad f = \text{constant}. \tag{5.141}$$

We can solve for $\sigma$ as in

$$\sigma = (f^2 - \boldsymbol{\pi}^2)^{1/2} \tag{5.142}$$

which is interpreted as a power series

$$\sigma = f\left(1 - \frac{1}{2}\frac{\boldsymbol{\pi}^2}{f^2} - \frac{1}{8}\frac{\boldsymbol{\pi}^4}{f^4} + \cdots\right). \tag{5.143}$$

**(a)** Show that the linear $\sigma$-model Lagrangian eqn (5.38), after eliminating the $\sigma$-field through eqn (5.142), is of the form

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\boldsymbol{\pi})^2 + \frac{1}{f^2 - \boldsymbol{\pi}^2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\right] + \bar{N}i\gamma^\mu\partial_\mu N \\ + g\bar{N}\left[\sqrt{f^2 - \boldsymbol{\pi}^2} + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right]N.\end{aligned} \tag{5.144}$$

**(b)** Calculate the scattering amplitudes, in the tree approximation, for the reactions:

**(i)** $N^a(p_1) + N^b(p_2) \to N^c(p_3) + N^d(p_4)$,
**(ii)** $\pi^i(k_1) + N^a(p_1) \to \pi^j(k_2) + N^b(p_2)$.

**Solution to Problem 5.7**

**(a)** We have the basic relation

$$\sigma = (f^2 - \boldsymbol{\pi}^2)^{1/2}. \tag{5.145}$$

To obtain an expression for $\partial_\mu\sigma$, we start by differentiating $\sigma^2 + \boldsymbol{\pi}^2 = f^2$ to obtain $\sigma\partial_\mu\sigma = -\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}$, which can be written as

$$\partial_\mu\sigma = -\frac{1}{\sigma}\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi} = -\frac{1}{(f^2 - \boldsymbol{\pi}^2)^{1/2}}\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}. \tag{5.146}$$

Substituting eqns (5.145) and (5.146) into the Lagrangian for the linear $\sigma$-model

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\boldsymbol{\pi})^2 + (\partial_\mu\sigma)^2\right] + \bar{N}i\gamma^\mu\partial_\mu N \\ + g\bar{N}(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})N + \frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2\end{aligned} \tag{5.147}$$

we get

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 + \frac{(\boldsymbol{\pi}\cdot\partial_\mu \boldsymbol{\pi})^2}{f^2 - \boldsymbol{\pi}^2}\right] + \bar{N} i\gamma^\mu \partial_\mu N$$
$$+ g\bar{N}[(f^2 - \boldsymbol{\pi}^2)^{1/2} + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}]N + \cdots \tag{5.148}$$
$$= \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 + \bar{N}(i\gamma^\mu\partial_\mu - m_N)N$$
$$+ g\bar{N}\left(-\frac{1}{2}\frac{\boldsymbol{\pi}^2}{f} + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right)N + \frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2}{2f^2} + \cdots . \tag{5.149}$$

**(b) (i)** $N^a(p_1) + N^b(p_2) \to N^c(p_3) + N^d(p_4)$.

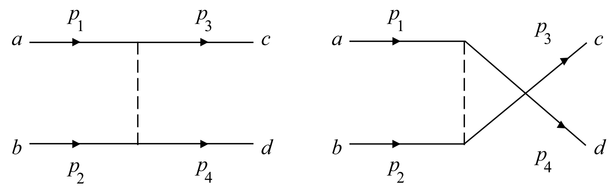



FIG. 5.1. $NN$ scatterings with pion exchanges in the $t$- and $u$-channels, respectively.

In the tree diagrams for these two processes, the basic pion nucleon vertex is the same as in the linear $\sigma$-model. For the first diagram with a pion exchanged in the $t$-channel:

$$\mathcal{T}_1 = g^2[\bar{u}(p_3)i\gamma_5(\tau^k)_{ca}u(p_1)]\frac{i}{t - m_\pi^2}[\bar{u}(p_4)i\gamma_5(\tau^k)_{db}u(p_2)] \tag{5.150}$$

where $t = (p_1 - p_3)^2$. Use the identity of CL-eqn (4.134),

$$\sum_k (\tau^k)_{ca}(\tau^k)_{db} = 2\left(\delta_{cb}\delta_{ad} - \frac{1}{2}\delta_{ca}\delta_{bd}\right), \tag{5.151}$$

to reduce the above amplitude to

$$\mathcal{T}_1 = 2g^2\left(\delta_{cb}\delta_{ad} - \frac{1}{2}\delta_{ca}\delta_{bd}\right)\bar{u}(p_3)i\gamma_5 u(p_1)\frac{i}{t - m_\pi^2}\bar{u}(p_4)i\gamma_5 u(p_2). \tag{5.152}$$

For the second diagram with the exchanged pion in the $u$ channel, we have

$$\mathcal{T}_2 = -2g^2\left(\delta_{db}\delta_{ac} - \frac{1}{2}\delta_{da}\delta_{bc}\right)\bar{u}(p_4)i\gamma_5 u(p_1)\frac{i}{u - m_\pi^2}\bar{u}(p_3)i\gamma_5 u(p_2) \tag{5.153}$$

where $u = (p_1 - p_4)^2$ and the extra minus sign in front is required by Fermi statistics.

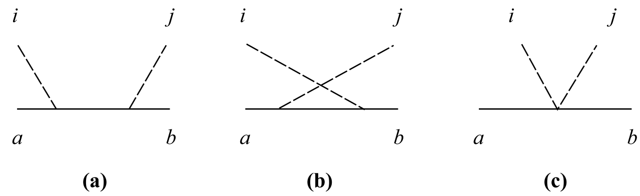



FIG. 5.2. Tree diagrams for $\pi N$ scatterings.

**(ii)** $\pi^i(k_1) + N^a(p_1) \to \pi^j(k_2) + N^b(p_2)$.
Here we have three tree diagrams

$$\mathcal{M}_1 = \sum_k \bar{u}(p_2) i g \gamma_5 (\tau^b)_{jk} \frac{i}{\not{p}_1 + \not{k}_1 - m_N} (\tau^a)_{ki} i g \gamma_5 u(p_1)$$

$$= \bar{u}(p_2) i g \gamma_5 \frac{i}{\not{p}_1 + \not{k}_1 - m_N} i g \gamma_5 u(p_1) (\tau^b \tau^a)_{ji}, \tag{5.154}$$

$$\mathcal{M}_2 = \bar{u}(p_2) i g \gamma_5 \frac{i}{\not{p}_1 - \not{k}_2 - m_N} i g \gamma_5 u(p_1) (\tau^a \tau^b)_{ji}, \tag{5.155}$$

$$\mathcal{M}_3 = \bar{u}(p_2) \frac{g}{f} u(p_1) \delta^{ab} \delta_{ji}. \tag{5.156}$$

## 5.8 Non-linear $\sigma$-model II

The constraint (5.141) can also be satisfied by parametrizations other than eqn (5.142) resulting in different versions of the non-linear $\sigma$-model, to be studied in this and the next problem.

**(a)** Show that the constraint $\sigma^2 + \boldsymbol{\pi}^2 = f^2$ can also be satisfied by the parametrization

$$\Sigma = \sigma + i \boldsymbol{\tau} \cdot \boldsymbol{\pi} = f \exp\left( i \frac{\boldsymbol{\tau} \cdot \boldsymbol{\phi}}{f} \right) \tag{5.157}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ are arbitrary functions.

**(b)** Show that the Lagrangian in this representation is of the form

$$\mathcal{L} = \frac{f2}{4} Tr \left( \partial_\mu \bar{\Sigma} \partial^\mu \bar{\Sigma}^\dagger \right) + \bar{N} i \gamma_\mu \partial^\mu N + g f (\bar{N}_L \bar{\Sigma} N_R + h.c.) \tag{5.158}$$

with

$$\bar{\Sigma} = \exp\left( i \frac{\boldsymbol{\tau} \cdot \boldsymbol{\phi}}{f} \right) \tag{5.159}$$

being the same as $\Sigma$ of eqn (5.157) except for the overall factor of $f$, and $\mathcal{L}$ is invariant under the transformations

$$\bar{\Sigma} \to \bar{\Sigma}' = L \bar{\Sigma} R^\dagger, \quad N_L \to N_L' = L N_L, \quad N_R \to N_R' = R N_R. \tag{5.160}$$

**(c)** Calculate the scattering amplitudes for the same reactions as those in Problem 5.7(b).

**Solution to Problem 5.8**

**(a)** From

$$\Sigma = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} = f\exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right) \tag{5.161}$$

$$\Sigma^{\dagger} = f\exp\left(-i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right) = \sigma - i\boldsymbol{\tau}\cdot\boldsymbol{\pi}, \tag{5.162}$$

we get

$$\Sigma^{\dagger}\Sigma = \sigma^2 + \boldsymbol{\pi}^2 = f^2\exp\left(-i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right)\exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right) = f^2. \tag{5.163}$$

**(b)** Since

$$(\partial_\mu\sigma)^2 + (\partial_\mu\boldsymbol{\pi})^2 = \frac{1}{2}Tr(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger) \tag{5.164}$$

and

$$\bar{N}[\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}]N = \bar{N}_L\Sigma N_R + \bar{N}_R\Sigma N_L \tag{5.165}$$

the Lagrangian can be written, using $\Sigma = f\bar{\Sigma}$, as

$$\mathcal{L}_2 = \frac{f2}{4}Tr\left(\partial_\mu\bar{\Sigma}\partial^\mu\bar{\Sigma}^\dagger\right) + \bar{N}i\gamma_\mu\partial^\mu N + gf(\bar{N}_L\bar{\Sigma}N_R + h.c.). \tag{5.166}$$

If we write $\bar{N}i\gamma_\mu\partial^\mu N = \bar{N}_L i\gamma_\mu\partial^\mu N_L + \bar{N}_R i\gamma_\mu\partial^\mu N_R$, it is easy to see that $\mathcal{L}_2$ is invariant under the transformations

$$\bar{\Sigma} \to L\bar{\Sigma}R^\dagger, \qquad N_L \to LN_L, \qquad N_R \to RN_R. \tag{5.167}$$

**(c)** Expanding $\bar{\Sigma}$ in powers of $\phi$

$$\frac{f2}{4}Tr\left(\partial_\mu\bar{\Sigma}\partial^\mu\bar{\Sigma}^\dagger\right) \to \frac{1}{2}(\partial_\mu\boldsymbol{\phi})^2 + \cdots. \tag{5.168}$$

$$\begin{aligned} gf(\bar{N}_L\bar{\Sigma}N_R + h.c.) &= gf\left[\bar{N}_L\left(1 + \frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f} - \frac{\boldsymbol{\phi}^2}{2f^2} + \cdots\right)N_R + h.c.\right] \\ &= m_N\bar{N}N + g\bar{N}(i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi})N \\ &\quad - \frac{g}{2f}\bar{N}(\boldsymbol{\phi}^2)N + \cdots. \end{aligned} \tag{5.169}$$

Comparing with the interaction given in Problem 5.7, it is clear that the $\boldsymbol{\phi}$ fields play the same role as the $\boldsymbol{\pi}$ fields as far as their couplings to $N(x)$ are concerned. Hence we recover the same scattering amplitudes as calculated in Problem 5.7.

**Remark.** Problems 5.7 and 5.8 are equivalent ways to realize the chiral SU(2) × SU(2) symmetry without the scalar field $\sigma(x)$. For example, in Problem 5.8 we have, under the axial transformations, $L = R^{\dagger} = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2}\right)$,

$$\bar{\Sigma} \to \bar{\Sigma}' = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2}\right)\exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right)\exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2}\right) = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}'}{f}\right).$$

For the case $|\boldsymbol{\beta}| \ll 1$, we can write

$$\left(1+\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2}\right)\exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right)\left(1+\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2}\right) = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}'}{f}\right)$$
$$= 1 + \frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}'}{f} + \cdots .$$

We can write the left-hand side as

$$\exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right) + \frac{i}{2}\left\{\boldsymbol{\tau}\cdot\boldsymbol{\beta}, \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right)\right\} = 1 + \frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f} + \frac{i\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2} + \cdots .$$

We then see that

$$\boldsymbol{\phi}' = \boldsymbol{\phi} + \frac{f}{2}\boldsymbol{\beta} + \cdots . \tag{5.170}$$

Clearly, the relation between $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$ is quite complicated and is, in general, non-linear. Thus the theories discussed in Problems 5.7 and 5.8 are referred to as non-linear realizations of the chiral symmetry. In Problem 5.9 we will study another non-linear representation.

## 5.9 Non-linear $\sigma$-model III

Suppose we redefine the fermion field of Problems 5.7 and 5.8 by a local axial transformation

$$N_L \to N_L' = LN_L, \qquad N_R \to N_R' = RN_R, \tag{5.171}$$

with

$$L = R^{\dagger} = \exp\left(-i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{2f}\right) \equiv \xi^{\dagger} \qquad \text{so that} \qquad \xi^2 = \bar{\Sigma}. \tag{5.172}$$

**(a)** Show that the Lagrangian can be written as

$$\mathcal{L}_3 = \frac{f2}{4}Tr\left(\partial_\mu\bar{\Sigma}\partial^\mu\bar{\Sigma}^{\dagger}\right) + \bar{N}_L' i\gamma_\mu\left[\partial_\mu + \xi^{\dagger}\partial_\mu\xi\right]N_L'$$
$$+ \bar{N}_R' i\gamma_\mu\left[\partial_\mu + \xi\partial_\mu\xi^{\dagger}\right]N_R' + gf(\bar{N_L}'N_R' + \bar{N}_R'N_L'). \tag{5.173}$$

**(b)** Calculate the scattering amplitudes, in the tree approximation, for the reactions:

**(i)** $N^a(p_1) + N^b(p_2) \to N^c(p_3) + N^d(p_4)$,

**(ii)** $\pi^i(k_1) + N^a(p_1) \to \pi^j(k_2) + N^b(p_2)$.

**Solution to Problem 5.9**

**(a)** From

$$N_R = R^\dagger N_R' = \xi^\dagger N_R', \qquad N_L = L^\dagger N_L' = \xi N_L', \tag{5.174}$$

we get

$$\bar{N}_L \bar{\Sigma} N_R + h.c. = \bar{N}_L' \xi^\dagger \bar{\Sigma} \xi^\dagger N_R' + h.c. = \bar{N}_L' N_R' + h.c. \tag{5.175}$$

where we have used

$$\xi^\dagger \bar{\Sigma} \xi^\dagger = \exp\left(-i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{2f}\right)\exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{f}\right)\exp\left(-i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{2f}\right) = 1. \tag{5.176}$$

From

$$\partial_\mu N_R = \left(\partial_\mu \xi^\dagger\right) N_R' + \xi^\dagger \partial_\mu N_R', \qquad \partial_\mu N_L = (\partial_\mu \xi) N_L' + \xi \partial_\mu N_L', \tag{5.177}$$

the Lagrangian (5.158) then becomes

$$\begin{aligned}\mathcal{L}_3 = {} & \frac{f2}{4} Tr\left(\partial_\mu \bar{\Sigma} \partial^\mu \bar{\Sigma}^\dagger\right) + m_N \bar{N}' N' \\ & + \bar{N}_L' i\gamma_\mu \left[\partial_\mu + \xi^\dagger \partial_\mu \xi\right] N_L' + \bar{N}_R' i\gamma_\mu \left[\partial_\mu + \xi \partial_\mu \xi^\dagger\right] N_R'. \end{aligned} \tag{5.178}$$

**Remark.** In this Lagrangian, the coupling of the Goldstone boson $\boldsymbol{\phi}$ to the $N$ fermion always contains a derivative,

$$\xi^\dagger \partial_\mu \xi = \left(1 - i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{2f} + \cdots\right) i\frac{\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}}{2f} \tag{5.179}$$

and

$$\begin{aligned}\mathcal{L}_3 = {} & \frac{f2}{4} Tr\left(\partial_\mu \bar{\Sigma} \partial^\mu \bar{\Sigma}^\dagger\right) + \bar{N}'(i\gamma_\mu\partial_\mu - m_N) N' \\ & + \bar{N}'\gamma_\mu\gamma_5 \left(\frac{\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}}{2f}\right) N' \\ & + \bar{N}'\gamma_\mu \frac{(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})}{(2f)^2} + \cdots . \end{aligned} \tag{5.180}$$

**(b) (i)** $N^i(p_1) + N^j(p_2) \to N^k(p_3) + N^l(p_4)$. The matrix element for the first diagram in Fig. 5.1 is given by

$$\begin{aligned}\mathcal{T}_1 = {} & [\bar{u}(p_3)\gamma_\mu\gamma_5(\tau^a)_{ki} u(p_1)] \frac{i(p_1 - p_3)^\mu}{2f} \\ & \times [\bar{u}(p_4)\gamma_\nu\gamma_5(\tau^a)_{lj} u(p_2)] \frac{i(p_4 - p_2)^\nu}{2f} \left(\frac{i}{t}\right). \end{aligned}$$

Using the Dirac equation we have

$$
\begin{aligned}
[\bar{u}(p_3)\gamma_\mu\gamma_5 u(p_1)]\frac{i(p_1-p_3)^\mu}{2f} &= \frac{-2m_N}{2f}[\bar{u}(p_3)i\gamma_5 u(p_1)] \\
&= -g[\bar{u}(p_3)i\gamma_5 u(p_1)].
\end{aligned}
$$

Similarly

$$
[\bar{u}(p_4)\gamma_\nu\gamma_5 u(p_2)]\frac{i(p_4-p_2)^\nu}{2f} = -g[\bar{u}(p_4)i\gamma_5 u(p_2)]. \tag{5.181}
$$

Thus we see that this is the same scattering amplitude as obtained in Problem 5.7 (hence also Problem 5.8). Clearly, this is also true for the other diagram for the $NN$ scattering.

**(ii)** $\pi^a(k_1)+N^i(p_1)\to\pi^b(k_2)+N^j(p_2)$. The matrix element for the diagram in Fig. 5.2(a) is

$$
\mathcal{M}'_1 = \bar{u}(p_2)\left(\frac{\not{k}_2\gamma_5}{2f}\right)\frac{i}{\not{p}_2+\not{k}_2-m_N}\left(\frac{\not{k}_1\gamma_5}{2f}\right)u(p_1)(\tau^b\tau^a)_{ji}. \tag{5.182}
$$

Write

$$
\not{k}_2\gamma_5 = [-\gamma_5(\not{p}_2+\not{k}_2-m_N)-(\not{p}_2-m_N)\gamma_5-2m_N\gamma_5]. \tag{5.183}
$$

Then

$$
\mathcal{M}'_1 = \frac{(\tau^b\tau^a)_{ji}}{(2f)^2}\bar{u}(p_2)\left[-\gamma_5\not{k}_1-2m_N\gamma_5\frac{i}{\not{p}_1+\not{k}_1-m_N}\not{k}_1\right]\gamma_5 u(p_1). \tag{5.184}
$$

Also using

$$
\not{k}_1\gamma_5 = [(\not{p}_1+\not{k}_1-m_N)\gamma_5+\gamma_5(\not{p}_1-m_N)+2m_N\gamma_5] \tag{5.185}
$$

for the second term, we get

$$
\begin{aligned}
\mathcal{M}'_1 = \frac{(\tau^b\tau^a)_{ji}}{(2f)^2}\Bigg\{&\bar{u}(p_2)(\not{k}_1-2m_N)u(p_1) \\
&-\bar{u}(p_2)(2m_N\gamma_5)\frac{i}{\not{p}_1+\not{k}_1-m_N}(2m_N\gamma_5)u(p_1)\Bigg\}.
\end{aligned} \tag{5.186}
$$

The second term is seen to be the same as the amplitude $M_1$ obtained in Problem 5.7 after using the relation $m_N = gf$. Similarly, for the diagram in Fig. 5.2(b)

$$
\begin{aligned}
\mathcal{M}'_2 = \frac{(\tau^a\tau^b)_{ji}}{(2f)^2}\Bigg\{&-\bar{u}(p_2)(\not{k}_2+2m_N)u(p_1) \\
&-\bar{u}(p_2)(2m_N\gamma_5)\frac{i}{\not{p}_2-\not{k}_1-m_N}(2m_N\gamma_5)u(p_1)\Bigg\}.
\end{aligned} \tag{5.187}
$$

For the seagull graph in Fig. 5.2(c) we have

$$\mathcal{M}_3' = \frac{1}{(2f)^2}\bar{u}(p_2)\left[(\tau^b\tau^a)_{ji}(-\not{k}_1) + (\tau^a\tau^b)_{ji}\not{k}_2\right]u(p_1). \tag{5.188}$$

The first terms in $\mathcal{M}_1'$ and $\mathcal{M}_2'$, combine with $\mathcal{M}_3'$ to give

$$\begin{aligned}\bar{\mathcal{M}}_3 &= \frac{1}{(2f)^2}\Big\{i\bar{u}(p_2)\Big[(\tau^b\tau^a)_{ji}(\not{k}_1 - 2m_N) - (\tau^a\tau^b)_{ji}(\not{k}_2 + 2m_N)\\ &\quad + (\tau^b\tau^a)_{ji}(-\not{k}_1) + (\tau^a\tau^b)_{ji}\not{k}_2\Big]u(p_1)\Big\}\\ &= \frac{1}{(2f)^2}\bar{u}(p_2)(-2m_N)(\tau^b\tau^a + \tau^a\tau^b)_{ij}u(p_1)\\ &= \frac{-g}{f}\bar{u}(p_2)\delta^{ab}\delta_{ij}u(p_1).\end{aligned} \tag{5.189}$$

This is precisely the amplitude $\mathcal{M}_3$ given in Problem 5.7.

We have demonstrated in Problems 5.7, 5.8, and 5.9 that these different ways to define the pion field all give the same *on-shell* S-matrix elements. The differences are in the off-shell behaviour. For example, in the realization of Problem 5.9, the pions couple to $N$ fields through derivative coupling and will vanish in the soft pion limit ($k_{i\mu} \to 0$). Since off-shell matrix elements are not measurable quantities, all these different realizations are physically equivalent. However, if one approximates some measurable quantities by some off-shell matrix element, then the difference in these realizations become significant. Which of these is the best approximation can only be decided by experiment and clearly will depend on the physical quantities of interest. For example, the realization given in Problem 5.9 seems to work quite well in the low-energy processes involving slow pions.

## 5.10 SSB by two scalars in the vector representation

**(a)** Show that a set of scalar fields $\boldsymbol{\phi}$ which transform as a vector representation in an O($n$) group can break the symmetry from O($n$) $\to$ O($n-1$).

**(b)** Show that for the case with two vectors in an O($n$) model, the spontaneous symmetry breaking (SSB) is at most

$$\mathrm{O}(n) \to \mathrm{O}(n-2). \tag{5.190}$$

**Solution to Problem 5.10**

**(a)** The $\boldsymbol{\phi}(x)$ fields belonging to a vector representation in O($n$) means that under O($n$) rotations we have

$$\phi_i \to \phi_i' = R_{ij}\phi_j \qquad \text{with} \qquad RR^T = R^T R = 1. \tag{5.191}$$

$R_{ij}$ are matrix elements that are real. The scalar product $\boldsymbol{\phi} \cdot \boldsymbol{\phi}$ is invariant under O($n$),

$$\phi_i' \phi_i' = R_{ij} R_{ik} \phi_j \phi_k = \delta_{jk} \phi_j \phi_k = \phi_j \phi_j. \tag{5.192}$$

The effective potential $V(\boldsymbol{\phi})$ which is O($n$) invariant can depend only on $\boldsymbol{\phi} \cdot \boldsymbol{\phi}$. For example,

$$V(\boldsymbol{\phi}) = -\frac{\mu^2}{2}(\boldsymbol{\phi} \cdot \boldsymbol{\phi}) + \frac{\lambda}{4}(\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2. \tag{5.193}$$

In other words, $V$ depends only on the magnitude $\phi = |\boldsymbol{\phi}|$ of the O($n$) vector,

$$V(\boldsymbol{\phi}) = V(\phi). \tag{5.194}$$

This means that the minimum of $V(\phi)$ depends only on $\phi$,

$$\phi^2 = \boldsymbol{\phi}^2 = \phi_1^2 + \phi_2^2 + \cdots + \phi_n^2 = v^2. \tag{5.195}$$

We can then choose the vector $\boldsymbol{\phi}$ to be in an arbitrary direction. For example, the choice

$$\boldsymbol{\phi} = (0, 0, \ldots, v) \tag{5.196}$$

will have the property that it is invariant under the rotation among the $n - 1$ coordinates $\phi_1, \phi_2, \ldots, \phi_{n-1}$,

$$R_{ij}' \phi_j = \phi_i, \qquad R_{ij}' = \begin{pmatrix} \bar{R} & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.197}$$

with $\bar{R}$, $(n-1) \times (n-1)$ orthogonal matrix. Thus the symmetry breaking is of the form

$$\mathrm{O}(n) \to \mathrm{O}(n-1). \tag{5.198}$$

**Remark.** In this case, the pattern of the symmetry breaking does not depend on the fact that $V(\boldsymbol{\phi})$ is a fourth-order polynomial in $\phi$.

**(b)** It is easy to see that the O($n$) invariant effective potential $V$ can depend only on the magnitudes of the vectors $\phi_1^2 = \boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_1$, $\phi_2^2 = \boldsymbol{\phi}_2 \cdot \boldsymbol{\phi}_2$, and the scalar product of the two vectors $\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2$, which can also be written as $\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_2 = \phi_1 \phi_2 \cos\theta$. The effective potential $V$ can then depend on three variables, $\phi_1$, $\phi_2$, and $\cos\theta$,

$$V = V(\phi_1, \phi_2, \cos\theta). \tag{5.199}$$

The minimization of $V$ determines the values of these three variables, $\phi_1 = v_1, \phi_2 = v_2, \cos\theta = \cos\alpha$. Clearly, these three variables define a plane, which can be taken to be the $(\phi_{n-1}, \phi_n)$ plane. Two vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ can have non-zero entries in the last two components. For example, one simple choice is

$$\boldsymbol{\phi}_1 = (0, 0, \ldots, v_1), \qquad \boldsymbol{\phi}_2 = v_2(0, 0, \ldots, \sin\alpha, \cos\alpha). \tag{5.200}$$

These configurations have the property that they are invariant under the rotations of the first $(n-2)$ components. The pattern of the symmetry breakings is then

$$\mathrm{O}(n) \to \mathrm{O}(n-2). \tag{5.201}$$

Note that it is possible that as a result of minimization, we have $\alpha = 0$ as the solution. (This can happen if $V$ depends on the even powers of $\cos\theta$ and the coefficient of the $\cos^2\theta$ term is negative.) This means that two vectors are parallel and the plane degenerates into a line. The symmetry breaking is then $O(n) \rightarrow O(n-1)$.

**Remark.** For the case of $k$ vectors in $O(n)$, the symmetry breaking is

$$O(n) \rightarrow O(n-k). \tag{5.202}$$

The generalization to unitary groups is straightforward and the result is that for the case of $k$ complex vectors in $SU(n)$ the symmetry breaking is

$$SU(n) \rightarrow SU(n-k), \qquad k < n. \tag{5.203}$$

# 6 Renormalization and symmetry

## 6.1 Path-integral derivation of axial anomaly

For the fermions, the generating functional can be written as a path integral of the form (see Fujikawa 1979)

$$Z[\eta, \bar{\eta}] = \int [d\psi] [d\bar{\psi}] \exp\left[ i \int \left( \mathcal{L} + \bar{\eta}\psi + \bar{\psi}\eta \right) \right]. \tag{6.1}$$

For simplicity, we will take the Lagrangian to have the form $\mathcal{L} = \bar{\psi} i \not{D} \psi$ with $D_\mu = \partial_\mu - igA_\mu$ being the covariant derivative and $A_\mu$ the $U(1)$ gauge field. One way to define the integration measure of the path integral is to expand $\psi$ and $\bar{\psi}$ in terms of a complete set of orthonormal functions, $\phi_n(x)$,

$$\psi(x) = \sum_n a_n \phi_n, \qquad \bar{\psi}(x) = \sum_n \phi_n^*(x) \bar{a}_n, \tag{6.2}$$

where

$$\int \phi_n^*(x) \phi_m(x) d^4x = \delta_{nm} \tag{6.3}$$

and define

$$\left[ d\bar{\psi} \right] \left[ d\bar{\psi} \right] = \prod_n da_n \prod_m d\bar{a}_m. \tag{6.4}$$

**(a)** Compute the Jacobian for the axial transformation

$$\psi \to \psi' = e^{i\alpha\gamma_5} \psi. \tag{6.5}$$

Show that for an infinitesimal $\alpha$, the Jacobian is of the form

$$J = I + i\alpha Tr(D) \qquad \text{where} \qquad Tr(D) = \sum_n \int d^4x \left( \phi_n^* \gamma_5 \phi_n \right). \tag{6.6}$$

**(b)** *TrD* is quite singular. If we take $\phi_n(x)$ to be the plane wave $\phi_n(x) = u(p, s)e^{-ipx}$, we get

$$\begin{aligned} TrD &= \int d^4x \, e^{ipx} \, u^\dagger(p, s) \gamma_5 u(p, s) e^{-ipx} \\ &= \delta^4(0) u^\dagger(p, s) \gamma_5 u(p, s) \end{aligned} \tag{6.7}$$

which is not well defined because $\delta^4(0) \to \infty$, while $u^\dagger(p, s)\gamma_5 u(p, s) \to 0$. It has been suggested by Fujikawa (1979) that we can regulate $Tr(D)$ by Gaussian cutoff

$$Tr(D) = \lim_{M\to\infty} \sum_n \int d^4x \left( \phi_n^* \gamma_5 \exp\left(-\frac{\lambda_n^2}{M^2}\right) \phi_n \right) \tag{6.8}$$

where $\lambda_n$ is the eigenvalue of the operator $i\not{D}$,

$$i\not{D}\chi_n = \lambda_n \chi_n. \tag{6.9}$$

Calculate $Tr(D)$ in the limit $M \to \infty$.

**(c)** Calculate the divergence of axial vector current $A_\mu$ as generated by the axial transformation (i.e. the anomaly equation).

**Solution to Problem 6.1**

**(a)** Expand the transformed field $\psi' = e^{i\alpha\gamma_5}\,\psi$, in a complete set of basis functions,

$$\psi' = \sum_n b_n \phi_n(x). \tag{6.10}$$

The coefficients of expansion can be projected out by using the orthogonality relation

$$\begin{aligned} b_n &= \int d^4x\, \phi_n^*(x)\psi'(x) = \int d^4x\, \phi_n^*(x) e^{i\gamma_5\alpha}\,\psi(x) \\ &= \int d^4x\, \phi_n^*(x) e^{i\gamma_5\alpha} \sum_m a_m \phi_m(x) = \sum_m C_{nm} a_m \end{aligned} \tag{6.11}$$

where

$$C_{nm} = \int d^4x\, \phi_n^*(x) e^{i\gamma_5\alpha}\,\phi_m(x). \tag{6.12}$$

Similarly,

$$\bar{\psi}' = \sum_n \bar{b}_n \phi_n^*(x), \qquad \bar{b}_n = \sum_m C_{nm}\bar{a}_m. \tag{6.13}$$

Thus the Jacobian of the transformation $(a_n, \bar{a}_n) \to (b_n, \bar{b}_n)$ is

$$J = (\det C)^2. \tag{6.14}$$

For infinitesimal $\alpha$, we have

$$C_{nm} \approx \delta_{nm} + i\alpha \int d^4x\, \phi_n^*(x)\gamma_5\phi_m(x) \tag{6.15}$$

or in matrix form

$$C \approx 1 + i\alpha D \qquad \text{with} \qquad D_{nm} = \int d^4x\, \phi_n^*(x)\gamma_5\phi_m(x). \tag{6.16}$$

Thus we get for the determinant:

$$\det C \approx \det(1 + i\alpha D) \approx 1 + i\alpha TrD \approx \exp(i\alpha TrD) \tag{6.17}$$

where

$$TrD = \sum_n \int d^4x \, \phi_n^*(x)\gamma_5\phi_n(x), \tag{6.18}$$

and we used the identity $\det\left(e^A\right) = e^{TrA}$. Thus we can write the Jacobian as an exponential:

$$J = (\det C)^2 \approx e^{2i\alpha TrD} = \exp\left\{2i\alpha \sum_n \int d^4x \, \phi_n^*(x)\gamma_5\phi_n(x)\right\}. \tag{6.19}$$

This means that the effect of an axial transformation can be included as an extra term in the Lagrangian,

$$\delta\mathcal{L}_\alpha = 2\alpha \sum_n \phi_n^*(x)\gamma_5\phi_n(x). \tag{6.20}$$

**(b)** Here we calculate the trace in eqn (6.18) with Gaussian regularization

$$TrD = \sum_n \int d^4x \, \phi_n^*(x)\gamma_5 \exp\left(-\frac{\lambda_n^2}{M^2}\right)\phi_n(x) \tag{6.21}$$

where $M$ is some regulator mass, and $\lambda_n$ is the eigenvalue of the operator $i\not{D}$,

$$i\not{D}\chi_n = \lambda_n\chi_n, \qquad D_\mu = \partial_\mu - igA_\mu. \tag{6.22}$$

For the special case of $g = 0$, we have $\lambda_n = \not{k}$ , and

$$\exp\left(-\frac{\lambda_n^2}{M^2}\right) = \exp\left(-\frac{k^2}{M^2}\right) \tag{6.23}$$

and the integral over $k$ is convergent. For the general case we choose $\phi_n(x)$ to be the eigenfunctions of the operator $i\not{D}$ and write $TrD$ as

$$TrD = \sum_n \int d^4x \, \phi_n^*(x)\gamma_5 \exp\left(\frac{\not{D}^2}{M^2}\right)\phi_n(x). \tag{6.24}$$

Since the trace is invariant under the change of basis (unitary transformation), we can now use the plane wave state

$$\phi_n(x) = e^{-ikx}, \quad \text{and} \quad \left(\sum_n \to \int \frac{d^4k}{(2\pi)^4}\right) \tag{6.25}$$

to compute the trace. Simple algebra gives the result

$$
\begin{aligned}
\not{D}\not{D} &= \gamma_\mu\gamma_\nu D^\mu D^\nu = \gamma_\mu\gamma_\nu \left(\tfrac{1}{2}[D^\mu, D^\nu] + \tfrac{1}{2}\{D^\mu, D^\nu\}\right) \\
&= \tfrac{1}{4}\{\gamma_\mu, \gamma_\nu\}\{D^\mu, D^\nu\} + \tfrac{1}{2}\gamma_\mu\gamma_\nu\,(-igF^{\mu\nu}) \\
&= \tfrac{1}{2}g_{\mu\nu}\{D^\mu, D^\nu\} - \frac{ig}{4}[\gamma_\mu, \gamma_\nu]F^{\mu\nu} = D^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu} \qquad (6.26)
\end{aligned}
$$

where

$$
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]. \qquad (6.27)
$$

Also,

$$
D^2 = \left(\partial_\mu - igA_\mu\right)(\partial^\mu - igA^\mu) = \partial^2 - 2igA^\mu\partial_\mu - ig\partial_\mu A^\mu - g^2A^\mu A_\mu
$$

$$
D^2\,e^{-ikx} = \left[-\left(k_\mu + gA_\mu\right)^2 - ig\partial_\mu A^\mu\right]e^{-ikx}. \qquad (6.28)
$$

Thus we have

$$
\exp\left(-\frac{D^2}{M^2}\right)e^{-ikx} = \exp\left[-\frac{\left(k_\mu + gA_\mu\right)^2}{M^2} - \frac{ig\partial_\mu A^\mu}{M^2}\right]e^{-ikx}. \qquad (6.29)
$$

Putting all these together, we get

$$
\begin{aligned}
TrD &= \int \frac{d^4k}{(2\pi)^4}\int d^4x\, Tr\left(\gamma_5 \exp\left(\frac{\not{D}^2}{M^2}\right)\right) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \\
&\quad \times Tr\left(\gamma_5 \exp\left[-\frac{\left(k_\mu + gA_\mu\right)^2}{M^2} - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\frac{1}{M^2} - \frac{ig}{M^2}\partial_\mu A^\mu\right]\right).
\end{aligned}
$$

Changing the integration variable, $k_\mu - gA_\mu = k'_\mu M$,

$$
TrD = \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2}\, Tr\left(\gamma_5 \exp\left[-\frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\frac{1}{M^2} - \frac{ig}{M^2}\partial_\mu A^\mu\right]\right).
$$

It is clear that the last term in exponential, not containing any $\gamma$-matrices, will not contribute as $Tr\gamma_5 = 0$. We can expand the exponential

$$
\begin{aligned}
\exp\left[-\frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}\frac{1}{M^2}\right] &= \exp\left[-\frac{ig}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\frac{1}{M^2}\right] \\
&= 1 - \frac{ig}{2}\gamma_\mu\gamma_\nu F^{\mu\nu}\frac{1}{M^2} \\
&\quad + \frac{1}{2}\left(\frac{ig}{2}\right)^2 \gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta F^{\mu\nu}F^{\alpha\beta}\frac{1}{M^4} + \cdots. \qquad (6.30)
\end{aligned}
$$

Only the first term and the $M^{-4}$ terms will survive as the $M^{-2}$ term will vanish after taking the trace, while the higher-order terms vanish in the limit $M \to \infty$.

Using the relation

$$Tr\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\right) = 4i\varepsilon_{\mu\nu\alpha\beta} \tag{6.31}$$

we get

$$TrD = -\frac{g^2}{8}\int d^4x \int \frac{d^4k}{(2\pi)^4} 4i\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}e^{-k^2}. \tag{6.32}$$

From

$$\int \frac{d^4k}{(2\pi)^4}e^{-k^2} = \frac{i}{16\pi^2} \tag{6.33}$$

we get

$$TrD = \frac{g^2}{32\pi^2}\int d^4x\, \varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{6.34}$$

**(c)** Thus the effective term in the Lagrangian is of the form

$$\delta\mathcal{L} = 2\alpha\frac{g^2}{32\pi^2}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{6.35}$$

Since the divergence of the axial vector current is just the coefficient of $\alpha(x)$ in $\delta\mathcal{L}$ under the axial transformation, we see that the Jacobian here will contribute to $\partial_\mu A^\mu$ as

$$\partial_\mu A^\mu = \frac{g^2}{16\pi^2}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{6.36}$$

Or, if we define

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \tag{6.37}$$

this can be written as

$$\partial_\mu A^\mu = \frac{g^2}{8\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{6.38}$$

which is just the axial anomaly equation.

## 6.2 Axial anomaly and $\eta \to \gamma\gamma$

The decay $\eta \to \gamma\gamma$ is very similar to $\pi^0 \to \gamma\gamma$. Suppose that the process also proceeds, like the case for $\pi^0$, through the axial anomaly. Parametrize the matrix elements for the decays, as in CL-eqns (6.61) and (6.63),

$$\mathcal{A}\left[P(q) \to \gamma\,(k_1, \varepsilon_1)\,\gamma\,(k_2, \varepsilon_2)\right] = \varepsilon_1^\mu\,(k_1)\,\varepsilon_2^\nu\,(k_2)\,i\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\Gamma_P\left(q^2\right) \tag{6.39}$$

where $P$ stands for either of the pseudoscalar mesons $\eta$ or $\pi^0$.

**(a)** If we assume $\eta$ is a pure octet, $\eta = \phi^8$, show that

$$\frac{\Gamma_\pi(0)}{\Gamma_\eta(0)} = \sqrt{3} \tag{6.40}$$

from the theory of anomaly.

**(b)** Show that the ratio of decay rates is given by

$$\frac{\Gamma\left(\pi^0 \to \gamma\gamma\right)}{\Gamma(\eta \to \gamma\gamma)} = \left(\frac{m_\pi}{m_\eta}\right)^3 \left[\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)}\right]^2. \tag{6.41}$$

Assume that

$$\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)} \approx \frac{\Gamma_\pi(0)}{\Gamma_\eta(0)}, \tag{6.42}$$

compute the decay ratio and compare it with the experimental results.

**Solution to Problem 6.2**

**(a)** From CL-eqns (6.69) and (6.72), we see that

$$\Gamma_\pi(0) = \frac{e^2}{4\pi^2 f_\pi} Tr\left(Q^2\lambda_3\right) \quad \text{and} \quad \Gamma_\eta(0) = \frac{e^2}{4\pi^2 f_\pi} Tr\left(Q^2\lambda_8\right).$$

Using

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

we get

$$\frac{\Gamma_\pi(0)}{\Gamma_\eta(0)} = \frac{Tr\left(\lambda_3 Q^2\right)}{Tr\left(\lambda_8 Q^2\right)} = \sqrt{3}. \tag{6.43}$$

**(b)** The amplitude is proportional to $f_\pi^{-1}$ and the decay rate is proportional to $f_\pi^{-2}$. This means that we need $m_P^3$ in the decay rates to get the right dimension,

$$\Gamma(P \to \gamma\gamma) \propto m_P^3 \Gamma_P\left(m_P^2\right). \tag{6.44}$$

Then we have

$$\frac{\Gamma\left(\pi^0 \to \gamma\gamma\right)}{\Gamma(\eta \to \gamma\gamma)} = \left(\frac{m_\pi}{m_\eta}\right)^3 \left[\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)}\right]^2. \tag{6.45}$$

If we assume

$$\frac{\Gamma_\pi\left(m_\pi^2\right)}{\Gamma_\eta\left(m_\eta^2\right)} \approx \frac{\Gamma_\pi(0)}{\Gamma_\eta(0)} = \sqrt{3}, \tag{6.46}$$

we get

$$\frac{\Gamma\left(\pi^0 \to \gamma\gamma\right)}{\Gamma(\eta \to \gamma\gamma)} = \left(\frac{m_\pi}{m_\eta}\right)^3 \times 3 = 0.045. \tag{6.47}$$

Experimentally, this ratio is about 0.0165. The discrepancy is probably due to the assumption (6.42). As $m_\pi^2 \approx 0.02\,\mathrm{GeV}^2$ which is quite close to 0, the approximation $\Gamma_\pi\left(m_\pi^2\right) \approx \Gamma_\pi(0)$ should be fairly good, while $m_\eta^2 \approx 0.3\,\mathrm{GeV}^2$ and $\Gamma_\eta\left(m_\eta^2\right) \approx \Gamma_\eta(0)$ is probably not a reliable approximation. Another possibility is that the $\eta$ meson does not transform as a pure member of the SU(3) octet.

## 6.3 Soft symmetry breaking and renormalizability

Consider the Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu\phi_1\right)^2 + \left(\partial_\mu\phi_2\right)^2\right] - \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{2}\left(\phi_1^2 + \phi_2^2\right)^2. \tag{6.48}$$

**(a)** Show that $\mathcal{L}$ is invariant under the transformation

$$\begin{aligned}\phi_1 &\to \phi_1' = \cos\theta\phi_1 + \sin\theta\phi_2\\ \phi_2 &\to \phi_2' = -\sin\theta\phi_1 + \cos\theta\phi_2.\end{aligned} \tag{6.49}$$

Use this symmetry to construct all possible counterterms.

**(b)** Suppose we add a symmetry-breaking term of the form

$$\mathcal{L}_{SB} = c\left(\phi_1^2 - \phi_2^2\right). \tag{6.50}$$

Construct all possible counterterms and show that $\mathcal{L} + \mathcal{L}_{SB}$ is still renormalizable.

### Solution to Problem 6.3

**(a)** This transformation is simply a rotation in the $(\phi_1, \phi_2)$ plane, and it leaves the combination $\phi_1^2 + \phi_2^2$ invariant just like the ordinary rotation on the plane. The superficial degree of divergence is given by

$$D = 4 - B_1 - B_2 \tag{6.51}$$

with $B_1$ and $B_2$ the numbers of external $\phi_1$ and $\phi_2$ lines. Note that owing to the symmetry $\phi_1 \to -\phi_1$, or $\phi_2 \to -\phi_2$, $B_1$ and $B_2$ have to be even.

(*i*) $B_1 = 2, B_2 = 0$, or $B_1 = 0,\ B_2 = 2$, implies that $D = 2$. We need the symmetric counterterms of the form,

$$\left(\partial_\mu\phi_1\right)^2 + \left(\partial_\mu\phi_2\right)^2, \qquad \left(\phi_1^2 + \phi_2^2\right). \tag{6.52}$$

(*ii*) $B_1 = 4,\ B_2 = 0$, or $B_1 = 0,\ B_2 = 4$, or $B_1 = B_2 = 2$ implies that $D = 0$. The counterterm which respects the symmetry is

$$\left(\phi_1^2 + \phi_2^2\right)^2. \tag{6.53}$$

**(b)** $\mathcal{L}_{SB} = c\left(\phi_1^2 - \phi_2^2\right)$. The index of divergence is $\delta = -2$, and the superficial degree of divergence is

$$D_{SB} = 4 - B_1 - B_2 - 2n_{SB} \tag{6.54}$$

where $n_{SB}$ is the number of times $\mathcal{L}_{SB}$ appears in the diagram. For diagrams with $n_{SB} = 0$, we need only the counterterms given in Part (a). For diagrams which contain one symmetry-breaking vertex, the degree of divergence is improved by 2:

$$D_{SB} = 2 - B_1 - B_2. \tag{6.55}$$

Thus only the two point functions are divergent: $B_1 = 2$, $B_2 = 0$ or $B_1 = 0$, $B_2 = 2$. The counterterms we need are $\phi_1^2$ and $\phi_2^2$. The combination $\left(\phi_1^2 + \phi_2^2\right)$ can be absorbed in the mass term $\frac{1}{2}\mu^2\left(\phi_1^2 + \phi_2^2\right)$ while the combination $\left(\phi_1^2 - \phi_2^2\right)$ can be absorbed in the symmetry-breaking term $\mathcal{L}_{SB} = c\left(\phi_1^2 - \phi_2^2\right)$. This implies that the theory with $\mathcal{L} + \mathcal{L}_{SB}$ is still renormalizable.

**Remark.** This is an explicit example which illustrates the *Symanzik theorem,* which states that if the symmetry breaking term has dimension $d_{SB} < 4$, we only need asymmetric counterterms with dimension $\leq d_{SB}$.

## 6.4 Calculation of the one-loop effective potential

As given in CL-eqn (6.121), the effective potential is of the form

$$V(\phi_c) = -\sum_n \frac{1}{n!}\Gamma^{(n)}(0, \ldots, 0)\,[\phi_c]^n \tag{6.56}$$

where $\Gamma^{(n)}(0, \ldots, 0)$ is the 1PI $n$-point Green's function in the momentum space and $\phi_c$ is the classical field. For simplicity of notation, replace $\phi_c$ by $\phi$. At the tree level we have

$$V = V_0(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{6.57}$$

which gives

$$\Gamma_0^{(2)}(0) = -\mu^2, \qquad \Gamma_0^{(4)}(0, \ldots,) = -\lambda. \tag{6.58}$$

Suppose we define a shifted field $\phi'$ by

$$\phi(x) = \phi'(x) + \omega \tag{6.59}$$

with $\omega$ an arbitrary constant, the Green's functions can then be expressed in terms of $\phi'$ and $\Gamma_\omega^{(n)}(0, \ldots, 0)$.

**(a)** Show that the effective potential has the property

$$V'(\omega) \equiv \frac{\partial V}{\partial \phi}\Big|_{\phi=\omega} = -\Gamma_\omega^{(1)}(0)\ , \tag{6.60}$$

where $\Gamma_\omega^{(1)}(0)$ is the one-point Green's function (the tadpole graph).

**(b)** Calculate $\Gamma_\omega^{(1)}(0)$ at the tree level and integrate it to get $V(\omega)$.

**(c)** Calculate $\Gamma_\omega^{(1)}(0)$ in one-loop and integrate it to get $V(\omega)$.

**Solution to Problem 6.4**

**(a)** The effective potential can be written in terms of the shifted field as

$$\begin{aligned} V(\phi) &= -\sum_n \frac{1}{n!}\Gamma^{(n)}(0,\ldots,0)[\phi]^n \\ &= -\sum_n \frac{1}{n!}\Gamma^{(n)}(0,\ldots,0)\left[\phi' + \omega\right]^n \\ &= -\sum_n \frac{1}{n!}\Gamma^{(n)}_\omega(0,\ldots,0)\left[\phi'\right]^n \\ &= -\sum_n \frac{1}{n!}\Gamma^{(n)}_\omega(0,\ldots,0)[\phi - \omega]^n. \end{aligned} \tag{6.61}$$

Thus it is clear that

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi=\omega} = -\Gamma^{(1)}_\omega(0) = V'(\omega). \tag{6.62}$$

This means that we can calculate the tadpole diagram's one-point function $\Gamma^{(1)}_\omega(0)$ in the shifted field $\phi'$, and integrate $\Gamma^{(1)}_\omega(0)$ over $\omega$ to get the effective potential $V(\phi)$.

**(b)** Expanding the potential in terms of the shifted field

$$\begin{aligned} V_0 &= \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 = \frac{\mu^2}{2}\left(\phi' + \omega\right)^2 + \frac{\lambda}{4!}\left(\phi' + \omega\right)^4 \\ &= V_0(\omega) + \left(\mu^2\omega + \frac{\lambda}{3!}\omega^3\right)\phi' + \cdots \end{aligned} \tag{6.63}$$

we get

$$\Gamma^{(1)}_\omega(0)_0 = -\left(\mu^2\omega + \frac{\lambda}{3!}\omega^3\right) = -\frac{\partial V_0}{\partial \omega}. \tag{6.64}$$

Integrating this relation we can get

$$V_0(\omega) = \int \frac{\partial V_0}{\partial \omega}\, d\omega = \int d\omega \left(\mu^2\omega + \frac{\lambda}{3!}\omega^3\right) = \frac{\mu^2}{2}\omega^2 + \frac{\lambda}{4!}\omega^4 \tag{6.65}$$

or

$$V_0(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{6.66}$$

This calculation, of course, is trivial. But it serves to illustrate the relation between $\Gamma^{(n)}_\omega$ and $\Gamma^{(n)}$.

**(c)** From

$$V_0(\phi) = V_0(\omega) + \left(\mu^2\omega + \frac{\lambda}{3!}\omega^3\right)\phi' + \left(\frac{\mu^2}{2} + \frac{\lambda\omega^2}{4}\right)\phi'^2 + \frac{\lambda\omega}{3!}\phi'^3 + \frac{\lambda}{4!}\phi'^4 \tag{6.67}$$

we can calculate the tadpole graph

$$\Gamma_\omega^{(1)}(0)_1 = \frac{i^2\lambda\omega}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - \left(\mu^2 + (\lambda\omega^2/2)\right)}. \tag{6.68}$$

Note that $\Gamma_\omega^{(1)}(0)$ is the 1PI one-point function and there is no propagator for the external line. Integrating this, we get

$$\begin{aligned} V_1(\omega) &= \int \Gamma_\omega^{(1)}(0)_1\, d\omega = -\tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\int \frac{\lambda\omega\, d\omega}{k^2 - \left(\mu^2 + (\lambda\omega^2/2)\right)} \\ &= -\tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left[k^2 - \left(\mu^2 + \frac{\lambda\omega^2}{2}\right) + i\varepsilon\right] + C \end{aligned} \tag{6.69}$$

where $C$ is independent of $\omega$. If we choose $C$ such that $V_1(\omega) = 0$ in the limit $\lambda = 0$, we have

$$V_1(\phi) = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left[1 - \frac{\lambda\phi^2/2}{k^2 - \mu^2 + i\varepsilon}\right] \tag{6.70}$$

which agrees with the result given in CL-eqn (6.139).

**Remark.** It is not hard to see that different choices of the $\omega$-independent $C$ correspond to different choices of counterterms and will not affect the properties of $V_{eff}(\phi)$ once the parameters in $V_{eff}(\phi)$ are fixed by the renormalization conditions.

# 7 The parton model and scaling

## 7.1 The Gottfried sum rule

In the parton model, if we assume that the proton quark sea has the same number of up and down quark pairs, i.e. in terms of the antiquark density $\bar{u}(x) = \bar{d}(x)$, show that

$$\int_0^1 \frac{dx}{x} \left[F_2^p(x) - F_2^n(x)\right] = \frac{1}{3}. \tag{7.1}$$

**Solution to Problem 7.1**

From the parton model, we have for the proton structure function

$$F_2^p(x) = x\left(\tfrac{4}{9}[\bar{u}+u] + \tfrac{1}{9}[\bar{d}+d] + \tfrac{1}{9}[\bar{s}+s]\right). \tag{7.2}$$

The neutron structure function can be obtained from the proton structure function by the substitutions $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$,

$$F_2^n(x) = x\left(\tfrac{1}{9}[\bar{u}+u] + \tfrac{4}{9}[\bar{d}+d] + \tfrac{1}{9}[\bar{s}+s]\right). \tag{7.3}$$

The proton and neutron difference is then

$$F_2^p(x) - F_2^n(x) = x\left[\tfrac{1}{3}(\bar{u}+u) - \tfrac{1}{3}(\bar{d}+d)\right]. \tag{7.4}$$

Since the total isospin of proton is $\frac{1}{2}$, we have the sum rule

$$\tfrac{1}{2} = \tfrac{1}{2}\int_0^1 [(u-\bar{u}) + (\bar{d}-d)]\,dx. \tag{7.5}$$

Combining eqns (7.4) and (7.5), we get

$$\begin{aligned}\int_0^1 \frac{dx}{x}[F_2^p(x) - F_2^n(x)] &= \tfrac{1}{3}\int_0^1 [(u-d) + (\bar{u}-\bar{d})]\,dx \\ &= \tfrac{1}{3} + \tfrac{2}{3}\int_0^1 (\bar{u}-\bar{d})\,dx. \end{aligned} \tag{7.6}$$

Thus if we assume $\bar{u} = \bar{d}$, the result is

$$\int_0^1 \frac{dx}{x}\left[F_2^p(x) - F_2^n(x)\right] = \frac{1}{3} \tag{7.7}$$

which is known as the Gottfried sum rule.

**Remark 1.** This sum rule can also be obtained with a weaker assumption,

$$\int_0^1 \bar{u}(x)\,dx = \int_0^1 \bar{d}(x)\,dx. \tag{7.8}$$

**Remark 2.** The assumption $\bar{u}(x) = \bar{d}(x)$ follows naturally from the simple picture of the quark pairs in the sea being created by the flavour-independent gluons and the up and down quarks having similar masses. However, for light mass quarks in the long-distance range, perturbative quantum chromodynamics (QCD) is not applicable. Since the proton is not an isotopic singlet, there is really no reason to expect its quark sea to be symmetric with respect to the $u$ and $d$ quark distributions.

## 7.2 Calculation of OPE Wilson coefficients

Consider the composite operator $J(x) = {:}\phi^2(x){:}$ in $\lambda\phi^4$ theory. Write the operator-product expansion (OPE) as

$$T(J(x)J(0)) = C_1(x)O_1(0) + C_2(x)O_2(0) + \cdots \tag{7.9}$$

where $C_i(x)$s are c-number functions and $O_i(0)$s are local operators.

**(a)** Write out the first three local operators, having the lowest dimensions, in terms of $\phi(0)$ and $\partial_\mu\phi(0)$ in this expansion.

**(b)** Define the Fourier transform by

$$\int d^4x\, e^{iqx}\, T(J(x)J(0)) = C_1(q)O_1(0) + C_2(q)O_2(0) + \cdots . \tag{7.10}$$

Use the Feynman rule to calculate the matrix element

$$T(p,q) = \int d^4x\, e^{iqx} \langle p|T(J(x)J(0))|p\rangle \tag{7.11}$$

to order $\lambda^0$. Then take the limit $q^2 \to -\infty$ to identify the coefficients $C_1(q), C_2(q), C_3(q)$.

**(c)** Draw Feynman diagrams which will contribute to $C_i(q)$ to order $\lambda$.

### Solution to Problem 7.2

**(a)** Since $J(x)$ is symmetric under $\phi \to -\phi$, we need to consider only operators which are even in $\phi$:

| dim | operators |
|---|---|
| 0 | $\mathbf{1}$ |
| 2 | ${:}\phi^2(0){:}$ |
| 4 | ${:}\phi^4(0){:}$, ${:}\partial_\nu\phi\partial_\mu\phi{:}$, ${:}\phi\partial_\nu\partial_\mu\phi{:}$ |

(7.12)

$$\begin{aligned} T(J(x)J(0)) = {} & C_1(x)\mathbf{1} + C_2(x){:}\phi^2(0){:} + C_3(x){:}\phi^4(0){:} \\ & + C_4^{\mu\nu}(x){:}\partial_\nu\phi\partial_\mu\phi{:} + C_5^{\mu\nu}(x){:}\phi\partial_\nu\partial_\mu\phi{:}\cdots . \end{aligned} \tag{7.13}$$

From dimensional analysis, we see that

$$C_1(x) \sim \frac{1}{x^4}, \qquad C_2(x) \sim \frac{1}{x^2}, \qquad C_{3,4}(x) \sim O(1). \tag{7.14}$$

**(b)** The first term in the OPE is a c-number and get its contribution from the disconnected graph, as shown in Fig. 7.1(a), with contribution

$$T^{(0)}(p,q) = \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - \mu^2} \frac{i}{(l+q)^2 - \mu^2}. \tag{7.15}$$

In coordinate space, this corresponds to

$$C_1(x) = [i\Delta_F(x)]^2 \qquad \text{where} \qquad \Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} e^{ik\cdot x} \frac{1}{k^2 - \mu^2 + i\varepsilon}.$$

For the connected graphs, there are two contributions to order $\lambda^0$; they are displayed in Fig. 7.1(b) and their matrix elements are

$$\begin{aligned} T^{(1)}(p,q) &= \frac{i}{(p+q)^2 - \mu^2} + \frac{i}{(p-q)^2 - \mu^2} \\ &= \frac{i}{q^2 + 2p\cdot q} + \frac{i}{q^2 - 2p\cdot q}. \end{aligned} \tag{7.16}$$

For $q^2$ large,

$$\frac{1}{q^2 + 2p\cdot q} = \frac{1}{q^2(1 + 2p\cdot q/q^2)} = \frac{1}{q^2}\left[1 - \frac{2p\cdot q}{q^2} + \left(\frac{2p\cdot q}{q^2}\right)^2 + \cdots\right]$$

and

$$T^{(1)}(p,q) = \frac{2i}{q^2} + \frac{8i(p\cdot q)^2}{q^4} + \cdots. \tag{7.17}$$

On the other hand, from the operator-product expansion,

$$\begin{aligned} \int d^4x\, e^{iqx}\, T(J(x)J(0)) &= C_1(q)\mathbf{1} + C_2(q){:}\phi^2(0){:} + C_3(q){:}\phi^4(0){:} \\ &\quad + C_4^{\mu\nu}(q){:}\partial_\nu\phi\partial_\mu\phi{:} + C_5^{\mu\nu}(q){:}\phi\partial_\nu\partial_\mu\phi{:}\cdots. \end{aligned} \tag{7.18}$$

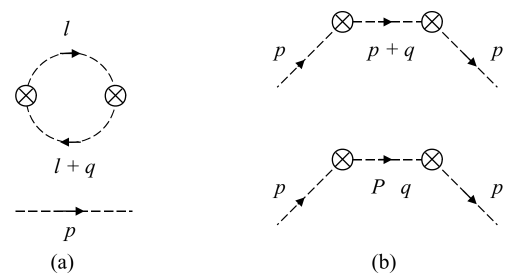



FIG. 7.1. Feynman diagrams for the operator-product expansion of two currents.

The matrix elements between one-particle states are given by

$$\int d^4x\, e^{iqx} \langle p|T(J(x)J(0))|p\rangle = C_2(q)\langle p|{:}\phi^2(0){:}|p\rangle + C_3(q)\langle p|{:}\phi^4(0){:}|p\rangle + C_4^{\mu\nu}(q)\langle p|{:}\partial_\nu\phi\partial_\mu\phi{:}|p\rangle + C_5^{\mu\nu}(q)\langle p|{:}\phi\partial_\nu\partial_\mu\phi{:}|p\rangle \cdots . \tag{7.19}$$

To order $\lambda^0$, we have free field theory and $\phi(x)$ can be expanded as

$$\phi(x) = \int \frac{d^3k}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} \left[a(k)\, e^{-ik\cdot x} + a^\dagger(k)\, e^{ik\cdot x}\right]. \tag{7.20}$$

Then the composite operator ${:}\phi^2(0){:}$ can be written as

$${:}\phi^2(0){:} = \int \frac{d^3k}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} \int \frac{d^3k'}{\left[(2\pi)^3 2\omega_{k'}\right]^{1/2}} \times [a(k)a(k') + a^\dagger(k)a^\dagger(k') + a^\dagger(k)a(k') + a^\dagger(k')a(k)]. \tag{7.21}$$

Using

$$|p\rangle = \left[(2\pi)^3 2\omega_p\right]^{1/2} a^\dagger(p)|0\rangle \qquad \text{and} \qquad [a(p), a^\dagger(k)] = \delta^3(p-k)$$

we get

$$\langle p|{:}\phi^2(0){:}|p\rangle = 2. \tag{7.22}$$

Similarly, a straightforward calculation gives

$$\langle p|{:}\phi^4(0){:}|p\rangle = 0 \tag{7.23}$$

because each term in ${:}\phi^4(0){:}$ will have at least two destruction operators on the right or two creation operators on the left. For the derivative of $\phi(x)$, we have

$$\partial_\mu\phi(x) = \int \frac{d^3k}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} (-ik_\mu)[a(k)\, e^{-ik\cdot x} + a^\dagger(k)\, e^{ik\cdot x}] \tag{7.24}$$

and

$${:}\partial_\mu\phi(0)\partial_\nu\phi(0){:} = \int \frac{d^3k}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} \int \frac{d^3k'}{\left[(2\pi)^3 2\omega_{k'}\right]^{1/2}} (-k_\mu k_\nu) \times [a(k)a(k') + a^\dagger(k)a^\dagger(k') - a^\dagger(k)a(k') - a^\dagger(k')a(k)], \tag{7.25}$$

$${:}\phi(0)\partial_\mu\partial_\nu\phi(0){:} = \int \frac{d^3k}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} \int \frac{d^3k'}{\left[(2\pi)^3 2\omega_{k'}\right]^{1/2}} (-k_\mu k_\nu) \times [a(k)a(k') - a^\dagger(k)a^\dagger(k') + a^\dagger(k)a(k') - a^\dagger(k')a(k)]. \tag{7.26}$$

From these we get

$$\langle p|{:}\partial_\nu\phi\partial_\mu\phi{:}|p\rangle = 2p_\mu p_\nu, \qquad \langle p|{:}\phi\partial_\nu\partial_\mu\phi{:}|p\rangle = 0. \tag{7.27}$$

**Remark 1.** The calculation of the matrix elements of the local operators is done in free field theory for illustrative purpose (to show how this can be carried out). For more general cases with interactions, these matrix elements are more complicated than those in free field theory. However, from Lorentz invariance it is not hard to see that the Lorentz structure of these matrix elements remains the same but the coefficients will have more complicated dependence on the coupling constant $\lambda$,

$$\langle p|{:}\phi^2(0){:}|p\rangle = a_1, \qquad \langle p|{:}\phi^4(0){:}|p\rangle = a_2, \tag{7.28}$$

$$\langle p|{:}\partial_\nu\phi\partial_\mu\phi{:}|p\rangle = a_3 p_\mu p_\nu, \qquad \langle p|{:}\phi\partial_\nu\partial_\mu\phi{:}|p\rangle = a_4 p_\mu p_\nu \tag{7.29}$$

where $a_1, a_2, a_3, a_4$ are constants which depend on $p^2 = m^2$ and the coupling constant $\lambda$. If the perturbation theory is applicable, we can expand these coefficients in powers of coupling constants $\lambda$,

$$a_i = a_i^{(0)} + \lambda a_i^{(1)} + \lambda^2 a_i^{(2)} + \cdots . \tag{7.30}$$

Our simple calculation gives the first terms in this expansion,

$$a_1^{(0)} = 2, \qquad a_2^{(0)} = 0, \qquad a_3^{(0)} = 2, \qquad a_4^{(0)} = 0. \tag{7.31}$$

To this order, we can use these matrix elements to read out the c-number coefficients from eqn (7.19),

$$C_2(q) = \frac{i}{q^2}, \qquad C_4^{\mu\nu}(q) = \frac{4iq^\mu q^\nu}{q^2}. \tag{7.32}$$

Note that because $a_4^{(0)} = 0$, we do not get any information on $C_5^{\mu\nu}(q)$ from this simple calculation. To get $C_5^{\mu\nu}(q)$ we need to use more complicated external states, e.g. two particles in the initial and final states.

**Remark 2.** The basic idea of calculating the Wilson coefficients is to use the fact that they are c-numbers and are process-independent. Thus we can choose some simple external states to simplify the matrix elements of the local operators and extract the Wilson coefficients.

**(c)**

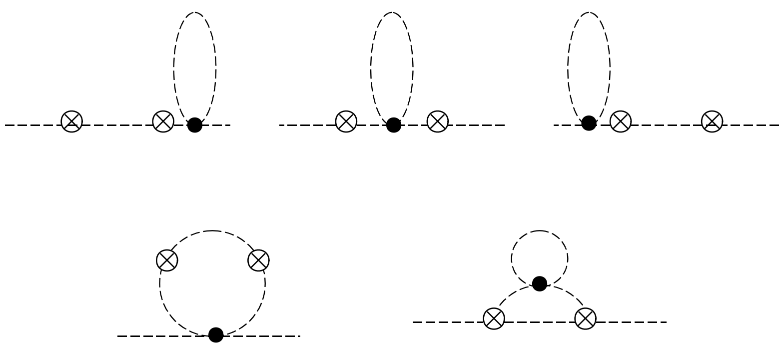


FIG. 7.2. Feynman diagrams for one-loop contribution.

## 7.3 $\sigma_{tot}(e^+e^- \to hadrons)$ and short-distance physics

Consider the process $e^+e^- \to hadrons$ through one-photon annihilation.

**(a)** Show that the total hadronic cross-section (summing over hadronic final states) can be written as

$$\sigma_{tot}(e^+e^- \to hadrons) = \frac{8\pi^2\alpha^2}{3(q^2)^2}\int d^4x\, e^{iq\cdot x}\langle 0|[J_\mu(x), J^\mu(0)]|0\rangle \tag{7.33}$$

where $J_\mu(x)$ is the electromagnetic current and $q^\mu$ the four-momentum of the intermediate photon.

**(b)** Suppose that $J_\mu(x)$ is made of free quarks:

$$J_\mu(x) = :\bar{q}(x)\gamma_\mu Q q(x): = \sum_i :\bar{q}_i(x)e_i\gamma_\mu q_i(x): \tag{7.34}$$

where $Q$ is the charge matrix and $i$ is the flavour index, calculate the commutator $\left[J_\mu(x), J^\mu(0)\right]$ and show that

$$\sigma_{tot}(e^+e^- \to hadrons) = \frac{4\pi\alpha^2}{3q^2}Tr(Q^2). \tag{7.35}$$

**(c)** Suppose that the current $J_\mu(x)$ is made out of free elementary scalar fields,

$$J_\mu(x) = i\sum_{i,j}\left[\phi_i^\dagger Q_{ij}\partial_\mu\phi_j - \partial_\mu\phi_i^\dagger Q_{ij}\phi_j\right]. \tag{7.36}$$

Calculate the commutator $\left[J_\mu(x), J^\mu(0)\right]$ and $\sigma_{tot}(e^+e^- \to hadrons)$.

### Solution to Problem 7.3

**(a)** The amplitude is given by

$$T = \bar{v}(-k')(-ie\gamma_\mu)u(k)\frac{(-ig^{\mu\nu})}{q^2}\langle n|J_\nu|0\rangle \tag{7.37}$$

and the cross-section is

$$d\sigma = \frac{1}{2|\mathbf{v}|}\frac{1}{2E}\frac{1}{2E'}(2\pi)^4\delta^4(k+k'-p_n)\prod_i^n\frac{d^3p_i}{(2\pi)^3 2p_{i0}}\left(\frac{1}{4}\sum_{spin}|T|^2\right).$$

The spin sum of $|T|^2$ is

$$\frac{1}{4}\sum_{spin}|T|^2 = -\frac{e^4}{4}Tr\left[\not{k}'\gamma_\mu\not{k}\gamma_\nu\right]\sum_n\langle 0|J^\nu|n\rangle\langle n|J^\mu|0\rangle\left(\frac{1}{q^2}\right)^2. \tag{7.38}$$

The leptonic tensor is

$$4l_{\mu\nu} = Tr\left[\not{k}'\gamma_\mu\not{k}\gamma_\nu\right] = 4\left(k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}k\cdot k'\right) \tag{7.39}$$

where we have used the approximation $k^2 = k'^2 = m_e^2 = 0$. The hadronic tensor is given by

$$\begin{aligned}\pi_{\mu\nu}(q) &= \int d^4x\, e^{iq\cdot x}\langle 0|[J_\mu(x), J_\nu(0)]|0\rangle \\ &= \sum_n \int d^4x\, e^{iq\cdot x}[\langle 0|J_\mu|n\rangle\langle n|J_\nu|0\rangle e^{-ip_n\cdot x} \\ &\quad -\langle 0|J_\nu|n\rangle\langle n|J_\mu|0\rangle e^{ip_n\cdot x}] \end{aligned} \tag{7.40}$$

As usual, the second term does not contribute for the case $q^0 > 0$ (because $p_n > 0$). Then we can write

$$\pi_{\mu\nu}(q) = \sum_n \prod_i^n \frac{d^3p_i}{(2\pi)^3 2p_{i0}}(2\pi)^4\delta^4(q - p_n)\langle 0|J_\mu|n\rangle\langle n|J_\nu|0\rangle \tag{7.41}$$

and the differential cross-section is

$$\begin{aligned} d\sigma &= \frac{1}{2E}\frac{1}{2E'}\sum_n (2\pi)^4\delta^4(q - p_n)\prod_i^n \frac{d^3p_i}{(2\pi)^3 2p_{i0}} e^4 l_{\mu\nu}\langle 0|J^\nu|n\rangle\langle n|J^\mu|0\rangle \\ &= \frac{1}{8EE'}\left(\frac{-e^4}{q^4}\right) l^{\mu\nu}\pi_{\mu\nu}. \end{aligned} \tag{7.42}$$

From Lorentz invariance and current conservation, we can write the hadronic tensor as

$$\pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\pi(q^2), \qquad \text{which gives} \qquad \pi_\mu^\mu = g^{\mu\nu}\pi_{\mu\nu} = 3q^2\pi(q^2).$$

Straightforward calculation yields

$$(q^2 g_{\mu\nu} - q_\mu q_\nu) l^{\mu\nu} = q^2(-2k\cdot k') - (2k\cdot qk'\cdot q - q^2 k\cdot k') = -q^4.$$

The total cross-section is then

$$\sigma_{tot} = \frac{1}{8EE'}\frac{e^4}{q^4} \times q^4\pi(q^2) = \frac{8\pi^2\alpha^2}{3q^4}\pi_\mu^\mu \tag{7.43}$$

or

$$\sigma_{tot} = \frac{8\pi^2\alpha^2}{3(q^2)^2}\int d^4x\, e^{iq\cdot x}\langle 0|[J_\mu(x), J^\mu(0)]|0\rangle. \tag{7.44}$$

**(b)** In CL-eqn (7.146), the c-number singularity in $T(J_\mu(x)J_\nu(0))$ is given by

$$C_{\mu\nu}(x) = \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4(x^2 - i\varepsilon)^4} = \left\{\frac{2}{3}\frac{g_{\mu\nu}}{(x^2 - i\varepsilon)^3} - \frac{1}{12}\partial_\mu\partial_\nu\left[\frac{1}{(x^2 - i\varepsilon)^2}\right]\right\} TrQ^2\frac{1}{\pi^4}.$$

From the substitution rule for going from T-product to the commutator

$$\frac{1}{(x^2 - i\varepsilon)^n} \to 2\pi i\varepsilon(x_0)\delta^{(n-1)}(x^2)\frac{1}{(n-1)!}(-1)^n \tag{7.45}$$

we get

$$C_{\mu\nu} = 2\pi i\left\{-\tfrac{1}{3}g_{\mu\nu}\varepsilon(x_0)\delta''(x^2) - \tfrac{1}{12}\partial_\mu\partial_\nu\left[\varepsilon(x_0)\delta'(x^2)\right]\right\}TrQ^2\frac{1}{\pi^4} \qquad (7.46)$$

and

$$g^{\mu\nu}C_{\mu\nu} = 2\pi i\left\{-\tfrac{4}{3}\varepsilon(x_0)\delta''(x^2) - \tfrac{1}{12}\partial^2\left[\varepsilon(x_0)\delta'(x^2)\right]\right\}TrQ^2\frac{1}{\pi^4}. \qquad (7.47)$$

The total cross-section is then

$$\sigma_{tot} = \frac{-8\pi^2\alpha^2}{3(q^2)^2}\left(\frac{iTrQ^2}{\pi^3}\right)\int d^4x\, e^{iq\cdot x}\left[\tfrac{8}{3}\varepsilon(x_0)\delta''(x^2) + \tfrac{1}{6}\partial^2\left[\varepsilon(x_0)\delta'(x^2)\right]\right].$$

Using the relation

$$\int d^4x\, e^{iq\cdot x}\delta^{(n)}(x^2)\varepsilon(x_0) = \frac{2^{2-2n}}{(n-1)!}\pi^2 i(q^2)^{n-1}\theta(q^2)\varepsilon(q_0) \qquad (7.48)$$

we get for the first term

$$\int d^4x\, e^{iq\cdot x}\delta''(x^2)\varepsilon(x_0) = \frac{\pi^2}{4}iq^2 \qquad (7.49)$$

and for the second term

$$\int d^4x\, e^{iq\cdot x}\partial^2\left[\delta'(x^2)\varepsilon(x_0)\right] = -q^2\int d^4x\, e^{iq\cdot x}\delta'(x^2)\varepsilon(x_0) = -q^2\pi^2 i.$$

The total cross-section is finally

$$\sigma_{tot} = \frac{-8\pi^2\alpha^2}{3(q^2)^2}\left(\frac{iTrQ^2}{\pi^3}\right)i\pi^2\left(\frac{2}{3}q^2 - \frac{1}{6}q^2\right) \qquad (7.50)$$

or

$$\sigma_{tot}(e^+e^- \to hadrons) = \frac{4\pi\alpha^2}{3q^2}Tr(Q^2). \qquad (7.51)$$

**(c)** Given the scalar field current operator

$$J_\mu(x) = i\sum_{i,j}\left[\phi_i^\dagger Q_{ij}\partial_\mu\phi_j - \partial_\mu\phi_i^\dagger Q_{ij}\phi_j\right] \qquad (7.52)$$

we have, after using Wick's theorem, for the c-number term in the operator product expansion,

$$\begin{aligned} &T\left(:\phi_i^\dagger(x)Q_{ij}\partial_\mu\phi_j(x)::\phi_k^\dagger(y)Q_{kl}\partial_\nu\phi_l(y):\right) \\ &= i\partial_\mu^x\Delta_F(x-y)i\partial_\nu^y\Delta_F(y-x)TrQ^2 \\ &= -i\partial_\mu^x\Delta_F(x-y)i\partial_\nu^x\Delta_F(y-x)TrQ^2 \\ &\xrightarrow[y=0]{} \partial_\mu^x\Delta_F(x)\partial_\nu^x\Delta_F(-x)TrQ^2. \end{aligned} \qquad (7.53)$$

Similarly,

$$T\left(:\phi_i^\dagger(x)Q_{ij}\partial_\mu\phi_j(x)::\partial_\nu\phi_k^\dagger(0)Q_{kl}\phi_l(0):\right) = [\partial_\nu\partial_\mu\Delta_F(x)]\Delta_F(-x)TrQ^2.$$

Using the relation

$$\partial_\nu\partial_\mu\Delta_F(x) = 2\left[\frac{-g_{\mu\nu}}{(x^2-i\varepsilon)^2} + \frac{4x_\mu x_\nu}{(x^2-i\varepsilon)^3}\right] \tag{7.54}$$

it is straightforward to get the OPE for the whole current

$$\begin{aligned} T(J_\mu(x)J_\nu(0)) &= 2\left\{\frac{4x_\mu x_\nu}{(x^2-i\varepsilon)^4} + \frac{2g_{\mu\nu}x^2}{(x^2-i\varepsilon)^4} - \frac{8x_\mu x_\nu}{(x^2-i\varepsilon)^4}\right\}\left(\frac{1}{4\pi^2}\right)^2 TrQ^2 \\ &= \frac{1}{4\pi^4}\frac{\left(g_{\mu\nu}x^2 - 2x_\mu x_\nu\right)}{(x^2-i\varepsilon)^4}TrQ^2. \end{aligned} \tag{7.55}$$

Comparing this to the case with the spin-1/2 constituent, we see that the scalar case has an extra factor of 1/4. Thus the total cross-section in this case is

$$\sigma_{tot}(e^+e^- \to hadrons) = \frac{\pi\alpha^2}{3q^2}Tr(Q^2). \tag{7.56}$$

**Remark.** Usually, the $e^+e^- \to$ *hadrons* cross-section is normalized to the $e^+e^- \to \mu^+\mu^-$ cross-section which measures the cross-section for the point-like particle and can be worked out for the tensor $\pi_{\mu\nu}(q)$ as follows. For a final state with $\mu^+(p_1)\mu^-(p_2)$, the tensor $\pi_{\mu\nu}(q)$ is given by

$$\begin{aligned} \pi_{\mu\nu}(q) = \int &\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_1}{(2\pi)^3 2E_1}(2\pi)^4\delta^4(q-p_1-p_2) \\ &\times \sum_{spin}\bar{v}(-p_1)\gamma_\mu u(p_2)\bar{u}(p_2)\gamma_\nu v(-p_1) \end{aligned} \tag{7.57}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^2}\int\frac{d^3p_1 d^3p_2}{4E_1E_2}\delta^4(q-p_1-p_2)(-4) \\ &\quad\times(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - p_1\cdot p_2 g_{\mu\nu}) \end{aligned} \tag{7.58}$$

and

$$\pi_\mu^\mu = \frac{2}{\pi^2}\int\frac{d^3p_1d^3p_2}{4E_1E_2}(p_1\cdot p_2)\delta^4(q-p_1-p_2). \tag{7.59}$$

Use the centre-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, we get

$$\begin{aligned} \pi_\mu^\mu(q) &= \frac{2}{\pi^2}\int\frac{d^3p_1}{4E^2}\left(\frac{q^2}{2}\right)\delta(q-2E) \\ &= \frac{q^2}{\pi^2}\int\frac{E^2dE}{4E^2}\delta(q-2E)4\pi \\ &= \frac{q^2}{2\pi}. \end{aligned}$$

Thus we get

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{8\pi^2\alpha^2}{3q^4}\pi^\mu_\mu = \frac{4\pi\alpha^2}{3q^2}. \tag{7.60}$$

For the case where the hadrons are made out of spin-1/2 quarks, we get

$$\frac{\sigma(e^+e^- \to \mathit{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = Tr(Q^2) = \sum_i Q_i^2 \tag{7.61}$$

One simple way to interpret this formula is to treat $\sigma_{tot}(e^+e^- \to \mathit{hadrons})$ as sum over $\sigma(e^+e^- \to q_i\bar{q}_i)$ where $q_i\bar{q}_i$ are treated as point-like spin-1/2 fermions.

## 7.4 OPE of two charged weak currents

The weak charged current at low energies is given by

$$J_\mu^W(x) = \bar{u}\gamma_\mu(1-\gamma_5)(d\cos\theta + s\sin\theta) = {:}\bar{q}(x)\gamma_\mu(1-\gamma_5)C_W q(x){:} \tag{7.62}$$

where $C_W$ is the weak coupling matrix

$$C_W = \begin{pmatrix} 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad q(x) = \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \tag{7.63}$$

Use Wick's theorem to work out the operator expansion for the product

$$T\left(J_\mu^W(x)J_\nu^{W\dagger}(0)\right) \tag{7.64}$$

in the free field theory.

**Solution to Problem 7.4**

Using Wick's theorem, we get

$$\begin{aligned} T&\left(J_\mu^W(x)J_\nu^{W\dagger}(0)\right) \\ &= T({:}\bar{q}(x)\gamma_\mu(1-\gamma_5)C_W q(x){:}{:}\bar{q}(0)\gamma_\nu(1-\gamma_5)C_W^\dagger q(0){:}) \\ &= Tr[iS_F(-x)\gamma_\mu(1-\gamma_5)iS_F(x)\gamma_\nu(1-\gamma_5)C_W C_W^\dagger] \\ &\quad + {:}\bar{q}(x)\gamma_\mu(1-\gamma_5)C_W iS_F(x)\gamma_\nu(1-\gamma_5)C_W^\dagger q(0){:} \\ &\quad + {:}\bar{q}(0)\gamma_\nu(1-\gamma_5)C_W^\dagger iS_F(-x)\gamma_\mu(1-\gamma_5)C_W q(x){:} \\ &\quad + {:}\bar{q}(x)\gamma_\mu(1-\gamma_5)C_W q(x)\bar{q}(0)\gamma_\nu(1-\gamma_5)C_W^\dagger q(0){:} \end{aligned} \tag{7.65}$$

Assuming the fermions are massless, which is valid for $|x| < \frac{1}{m}$, we have

$$S_F(x) \simeq i\gamma\cdot\partial\left[\frac{i}{4\pi^2}\frac{1}{(x^2 - i\varepsilon)}\right] = \frac{1}{2\pi^2}\frac{\not{x}}{(x^2 - i\varepsilon)^2}. \tag{7.66}$$

The first term is then

$$\left(\frac{1}{2\pi^2}\right) Tr\left(C_W C_W^\dagger\right) Tr\left[-\not{x}\,\gamma_\mu(1-\gamma_5)\not{x}\,\gamma_\nu(1-\gamma_5)\right]\frac{1}{(x^2-i\varepsilon)^2}$$
$$= 2Tr\left(C_W C_W^\dagger\right)\frac{\left(x^2 g_{\mu\nu}-x_\mu x_\nu\right)}{\pi^4(x^2-i\varepsilon)^4}. \qquad (7.67)$$

The second term is

$$:\bar{q}(x)\gamma_\mu(1-\gamma_5)C_W C_W^\dagger\left[\frac{1}{2\pi^2}\frac{i\not{x}}{(x^2-i\varepsilon)^2}\right]\gamma_\nu(1-\gamma_5)q(0):$$
$$=\frac{ix^\alpha}{\pi^2(x^2-i\varepsilon)^2}\left\{(s_{\mu\alpha\nu\beta}+i\varepsilon_{\mu\alpha\nu\beta}):\bar{q}(x)\gamma^\beta(1-\gamma_5)C_W C_W^\dagger q(0):\right\} \qquad (7.68)$$

where we have used the relation [see CL-eqn (7.145)]

$$\gamma_\mu\not{x}\,\gamma_\nu(1-\gamma_5)=i\left(s_{\mu\alpha\nu\beta}+i\varepsilon_{\mu\alpha\nu\beta}\right)\gamma^\beta(1-\gamma_5)x^\alpha. \qquad (7.69)$$

Similarly, the third term is

$$\frac{-ix^\alpha}{\pi^2(x^2-i\varepsilon)^2}\left\{(s_{\mu\alpha\nu\beta}+i\varepsilon_{\mu\alpha\nu\beta}):\bar{q}(0)\gamma^\beta(1-\gamma_5)C_W^\dagger C_W q(x):\right\}. \qquad (7.70)$$

Thus we can write the operator product expansion in the free field theory as

$$T\left(J_\mu^W(x)J_\nu^{W\dagger}(0)\right)$$
$$= 2Tr\left(C_W C_W^\dagger\right)\frac{\left(x^2 g_{\mu\nu}-x_\mu x_\nu\right)}{\pi^4(x^2-i\varepsilon)^4}+\frac{ix^\alpha}{\pi^2(x^2-i\varepsilon)^2}\left\{(s_{\mu\alpha\nu\beta}+i\varepsilon_{\mu\alpha\nu\beta})\right.$$
$$\left.\times\left[:\bar{q}(x)\gamma^\beta(1-\gamma_5)C_W C_W^\dagger q(0)::-\bar{q}(0)\gamma^\beta(1-\gamma_5)C_W^\dagger C_W q(x):\right]\right\}$$
$$+:\bar{q}(x)\gamma_\mu(1-\gamma_5)C_W q(x)\bar{q}(0)\gamma_\nu(1-\gamma_5)C_W^\dagger q(0):. \qquad (7.71)$$

## 7.5 The total decay rate of the $W$-boson

Use the operator-product expansion derived in the last problem to calculate the total decay rate for $W^\pm \to$ *hadrons*.

**Solution to Problem 7.5**

The amplitude for $W^\pm \to$ *hadrons* decay is given by

$$T=\frac{g}{2\sqrt{2}}\varepsilon^\mu(k)\langle n|J_\mu^W|0\rangle \qquad (7.72)$$

where $J_\mu^W$ is the weak charged current. The total decay rate is

$$\Gamma=\frac{1}{2M_W}\sum_n\int(2\pi)^4\delta^4(k-p_n)\prod_{i=1}^n\frac{d^3p_i}{(2\pi)^3 2p_{i0}}\left(\frac{1}{3}\sum_{spin}|T|^2\right). \qquad (7.73)$$

The spin sum is

$$\frac{1}{3}\sum_{spin}|T|^2 = \frac{g^2}{24}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}\right)\sum_n \langle 0|J_\nu^{W\dagger}|n\rangle\langle n|J_\mu^W|0\rangle. \tag{7.74}$$

Define the weak hadronic tensor by

$$\begin{aligned}
\pi_{\mu\nu}^W(k) &= \int d^4x\, e^{ik\cdot x}\langle 0|\left[J_\mu^{W\dagger}(x), J_\nu^W(0)\right]|0\rangle \\
&= \sum_n \int d^4x\, e^{ik\cdot x}[\langle 0|J_\mu^{W\dagger}|n\rangle\langle n|J_\nu^W|0\rangle\, e^{-ip_n\cdot x} \\
&\quad -\langle 0|J_\nu^W|n\rangle\langle n|J_\mu^{W\dagger}|0\rangle\, e^{ip_n\cdot x}] \\
&= \sum_n \prod_{i=1}^n \frac{d^3p_i}{(2\pi)^3 2p_{i0}}(2\pi)^4\delta^4(k - p_1 - \cdots - p_n)\,\langle 0|J_\mu^{W\dagger}|n\rangle\langle n|J_\nu^W|0\rangle
\end{aligned} \tag{7.75}$$

where we have used the fact that $k_0 > 0$ to eliminate the term proportional to $e^{ip_n\cdot x}$. The total decay rate can then be written as

$$\Gamma = \frac{g^2}{48M_W}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}\right)\pi_{\mu\nu}^W(k). \tag{7.76}$$

From the Lorentz invariance, we can write

$$\pi_{\mu\nu}^W(k) = -\pi_1^W k^2 g_{\mu\nu} + \pi_2^W k_\mu k_\nu \tag{7.77}$$

and the decay rate is then

$$\Gamma = \frac{g^2}{16}M_W^2\pi_1^W. \tag{7.78}$$

The weak charged current is generally of the form

$$J_\mu^W = \bar{u}\gamma_\mu(1-\gamma_5)U_{ui}d_i + \bar{c}\gamma_\mu(1-\gamma_5)U_{ci}d_i \tag{7.79}$$

where $U_{ui}$s are the Kobayashi–Maskawa (KM) matrix elements. It is easy to see that we can simply replace

$$Tr\left(C_W C_W^\dagger\right) \to 3\sum_{i=d,b,s}\left(|U_{ui}|^2 + |U_{ci}|^2\right) \tag{7.80}$$

in the operator product expansion for the weak current worked out in the last problem. The factor of 3 is due to the summation over colour. Then following the same calculation as in the case for the electromagnetic current we get

$$\frac{\Gamma(W^\pm \to hadrons)}{\Gamma(W^\pm \to e^\pm\nu)} = 3\sum_{i=d,b,s}\left(|U_{ui}|^2 + |U_{ci}|^2\right). \tag{7.81}$$

# 8 Gauge symmetries

## 8.1 The gauge field in tensor notation

The SU$(n)$ groups consist of $n \times n$ unitary unimodular matrices, $U^\dagger U = UU^\dagger = 1$. For infinitesimal group transformations, we can write

$$U_{jk} = \delta_{jk} + i\epsilon_{jk} \tag{8.1}$$

where $\epsilon$ is hermitian

$$\epsilon_{jk} = \epsilon^*_{kj}. \tag{8.2}$$

It is more convenient to use upper and lower indices so that

$$\epsilon_{jk} \equiv \epsilon_j{}^k \tag{8.3}$$

and complex conjugation interchanges upper and lower indices

$$\epsilon_j{}^k = \left(\epsilon^j{}_k\right)^*. \tag{8.4}$$

The hermiticity relation (8.2) can then be written as

$$\epsilon_j{}^k = \epsilon^k{}_j. \tag{8.5}$$

This means that the ordering of the upper and lower indices contains non-trivial information.

The $n$-dimensional vector $\phi_i$ and its complex conjugate $\phi^i$ have the following infinitesimal transformation law,

$$\phi_i \to \phi_i + i\varepsilon_i{}^l\phi_l, \qquad \phi^i \to \phi^i - i\varepsilon^i{}_k\phi^k, \tag{8.6}$$

where

$$\varepsilon_i{}^k = \left(\varepsilon^i{}_k\right)^* \qquad \text{and} \qquad \phi^i = (\phi_i)^*. \tag{8.7}$$

For the fields in the adjoint representation $\phi_i{}^j$, we have

$$\phi_i{}^j \to \phi_i{}^j + i\varepsilon_i{}^l\phi_l{}^j - i\varepsilon^j{}_k\phi_i{}^k. \tag{8.8}$$

**(a)** Construct the covariant derivatives for $\phi_i$ and $\phi^i$, respectively. Show that the transformation law for the gauge bosons is

$$W_{\mu i}{}^j \to W'_{\mu i}{}^j = W_{\mu i}{}^j + i\varepsilon_i{}^l W_{\mu l}{}^j - i\varepsilon^j{}_k W_{\mu i}{}^k - \frac{1}{g}\partial_\mu\varepsilon_i{}^j. \tag{8.9}$$

**(b)** Construct the field strength tensor $F_{\mu\nu j}{}^i$ for the gauge fields $W_{\mu i}{}^j$.

**(c)** Construct the covariant derivative for scalar fields in the adjoint representation.

**Solution to Problem 8.1**

**(a)** The covariant derivative acting on $\phi_i$ should be of the form

$$(D_\mu\phi)_i = \partial_\mu\phi_i + ig(W_\mu)_i{}^k\phi_k \tag{8.10}$$

with $(D_\mu\phi)_i$ having the same transformation as $\phi_i$,

$$(D_\mu\phi)'_i = (D_\mu\phi)_i + i\varepsilon_i{}^l(D_\mu\phi)_l. \tag{8.11}$$

The left-hand side (LHS) and right-hand side (RHS) of this equation can be written out as

$$\begin{aligned}\text{LHS} &= \partial_\mu\phi'_i + igW'_{\mu i}{}^l\phi'_l = \partial_\mu\left(\phi_i + i\varepsilon_i^l\phi_l\right) + igW'^l_{\mu i}\left(\phi_l + i\varepsilon_l{}^k\phi_k\right)\\ &= \partial_\mu\phi_i + i\partial_\mu\varepsilon_i{}^l\phi_l + i\varepsilon_i^l\partial_\mu\phi_l + igW'^l_{\mu i}\left(\phi_l + i\varepsilon_l{}^k\phi_k\right)\end{aligned} \tag{8.12}$$

$$\text{RHS} = \partial_\mu\phi_i + igW^l_{\mu i}\phi_l + i\varepsilon_i{}^l\left(\partial_\mu\phi_l + igW^k_{\mu l}\phi_k\right). \tag{8.13}$$

By equating these two expressions, we get

$$i\partial_\mu\varepsilon_i{}^l\phi_l + igW'_{\mu i}{}^l\left(\phi_l + i\varepsilon_l{}^k\phi_k\right) = igW_{\mu i}{}^l\phi_l + i^2\varepsilon_i^l gW^k_{\mu l}\phi_k. \tag{8.14}$$

Since $\phi_l$s are arbitrary, we can cancel them on both sides,

$$i\partial_\mu\varepsilon_i{}^l + igW'_{\mu i}{}^k\left(\delta_k{}^l + i\varepsilon_k{}^l\right) = ig\left(W^l_{\mu i} + i\varepsilon_i{}^kW^l_{\mu k}\right). \tag{8.15}$$

Multiply both sides by $\left(\delta_l^j - i\varepsilon_l{}^j\right)$ and drop the terms of order $\varepsilon^2$, we get

$$W'_{\mu i}{}^j = W_{\mu i}{}^j + i\varepsilon_i{}^lW_{\mu l}{}^j - i\varepsilon_k{}^jW_{\mu i}{}^k - \frac{1}{g}\partial_\mu\varepsilon_i{}^j. \tag{8.16}$$

**(b)** To find the field strength tensor $F_{\mu\nu i}{}^j$ we calculate the combination $(D_\mu D_\nu - D_\nu D_\mu)$.

$$\begin{aligned}(D_\mu D_\nu\phi)_i &= [D_\mu(D_\nu\phi)]_i = \partial_\mu(D_\nu\phi)_i + igW_{\mu i}^{\ l}(D_\nu\phi)_l\\ &= \partial_\mu\left(\partial_\nu\phi_i + igW^k_{\nu i}\phi_k\right) + igW_{\mu i}^{\ l}\left(\partial_\nu\phi_l + igW^k_{\nu l}\phi_k\right)\\ &= \partial_\mu\partial_\nu\phi_i + ig\partial_\mu W^k_{\nu i}\phi_k + ig\left(W_{\nu i}{}^k\partial_\mu\phi_k + W^k_{\mu i}\partial_\nu\phi_k\right)\\ &\quad + (ig)^2W^l_{\mu i}W^k_{\nu l}\phi_k.\end{aligned} \tag{8.17}$$

Then we have for the antisymmetrized combination,

$$(D_\mu D_\nu - D_\nu D_\mu)\phi_i = ig\left[\partial_\mu W^k_{\nu i} - \partial_\nu W^k_{\mu i} + ig\left(W_{\mu i}{}^lW_{\nu l}{}^k - W_{\nu i}{}^lW_{\mu l}{}^k\right)\right]\phi_k. \tag{8.18}$$

This means that the field strength tensor should be of the form

$$F_{\mu\nu i}{}^k = \partial_\mu W_{\nu i}{}^k - \partial_\nu W_{\mu i}{}^k + ig\left(W_{\mu i}{}^lW_{\nu l}{}^k - W_{\nu i}{}^lW_{\mu l}{}^k\right). \tag{8.19}$$

**(c)** From the global transformation, it is easy to see that the combination $W\phi \equiv \phi_W$ which transforms as an adjoint representation is given by

$$(\phi_W)_i{}^j = W_{\mu i}{}^k \phi_k{}^j - W_{\mu k}{}^j \phi_i{}^k. \tag{8.20}$$

This can be seen as follows. Under the global transformation, we have

$$\begin{aligned}
\left(\phi'_W\right)_i{}^j &= W'_{\mu i}{}^k \phi'_k{}^j - W'_{\mu k}{}^j \phi'_i{}^k \\
&= \left(W_{\mu i}{}^k + i\varepsilon_i{}^l W_{\mu l}{}^k - i\varepsilon_m{}^k W_{\mu i}{}^m\right)\left(\phi_k{}^j + i\varepsilon_k^m \phi_m{}^j - i\varepsilon_n{}^j \phi_k{}^n\right) \\
&\quad -\left(W_{\mu k}{}^j + i\varepsilon_k{}^l W_{\mu l}{}^j - i\varepsilon_m{}^j W_{\mu k}{}^m\right)\left(\phi_i{}^k + i\varepsilon_i{}^m \phi_m{}^k - i\varepsilon_n{}^k \phi_i{}^n\right) \\
&= W_{\mu i}{}^k \phi_k{}^j - W_{\mu k}{}^j \phi_i{}^k + iW_{\mu i}{}^k\left(-\varepsilon_n{}^j \phi_k{}^n\right) + i\left(\varepsilon_i{}^l W_{\mu l}{}^k\right)\phi_k{}^j \\
&\quad - W_{\mu k}{}^j\left(i\varepsilon_i{}^m \phi_m{}^k\right) - \left(i\varepsilon_k{}^l W_{\mu l}{}^j\right)\phi_i{}^k \\
&= (\phi_W)_i{}^j + i\varepsilon_i{}^l\,(\phi_W)_l{}^j - i\varepsilon_m{}^j (\phi_W)_i{}^m.
\end{aligned} \tag{8.21}$$

Therefore, the covariant derivative should be of the form

$$(D_\mu \phi)_i{}^j = \partial_\mu \phi_i{}^j + ig\left(W_{\mu i}^k \phi_k^j - W_{\mu k}^j \phi_i^k\right). \tag{8.22}$$

**Remark.** If we expand $\phi_i{}^j$ and $W_{\mu i}^k$ in terms of the hermitian traceless $n \times n$ matrices $\lambda^a$,

$$\phi_i{}^j = (\lambda^a)_i{}^j \phi_a, \qquad W_{\mu i}{}^j = (\lambda^b)_i{}^j W_\mu^b, \tag{8.23}$$

we have

$$\begin{aligned}
(D_\mu \phi)_i^j &= (\lambda^a)_i{}^j D_\mu \phi_a = \partial_\mu \phi_a (\lambda^a)_i{}^j \\
&\qquad + ig\left(W_{a\mu}(\lambda^a)_i{}^k (\lambda^b)_k{}^j \phi_b - W_{a\mu}(\lambda^b)_i{}^k (\lambda^a)_k{}^j \phi_b\right) \\
&= \partial_\mu \phi_a (\lambda^a)_i{}^j + ig\left([\lambda^a, \lambda^b]_i{}^j W_{a\mu}\phi_b\right) \\
&= \partial_\mu \phi_a (\lambda^a)_i{}^j + ig\left(if^{abc}(\lambda^c)_i{}^j W_{a\mu}\phi_b\right)
\end{aligned} \tag{8.24}$$

and

$$D_\mu \phi_a = \partial_\mu \phi_a - gf^{abc} W_{b\mu}\phi_c. \tag{8.25}$$

Or, if we write $if^{abc} = (t^b)_{ac}$, the adjoint representation matrix, then

$$D_\mu \phi_a = \partial_\mu \phi_a + ig(t^b)_{ac} W_{b\mu}\phi_c \tag{8.26}$$

which is the standard form for the covariant derivative.

## 8.2 Gauge field and geometry

Under the local gauge transformations, fields at different points transform differently.

$$\psi(x) \to U(x)\psi(x), \qquad \psi(y) \to U(y)\psi(y) \tag{8.27}$$

with $U(x) \neq U(y)$. Thus the usual derivative, being proportional to the difference of fields at different points,

$$\partial\psi(x) \propto [\psi(x+dx) - \psi(x)], \tag{8.28}$$

does not have a simple transformation property because $U(x+dx) \neq U(x)$. Suppose we introduce the gauge fields $A_\mu$ such that we can define

$$\tilde{\psi}(x+dx) = \psi(x+dx) + A_\mu(x)\psi(x)\,dx^\mu \tag{8.29}$$

so that $\tilde{\psi}(x+dx)$ transforms the same way as $\psi(x)$, i.e.

$$\tilde{\psi}(x+dx) \to U(x)\tilde{\psi}(x+dx). \tag{8.30}$$

**(a)** Show that if we define the covariant derivative by [see CL-Section 8.2 for discussion of the concept of a covariant derivative in connection with parallel transport of a field]

$$\tilde{\psi}(x+dx) - \psi(x) \equiv D_\mu\psi\,dx^\mu \tag{8.31}$$

then

$$D_\mu\psi = (\partial_\mu + A_\mu)\psi. \tag{8.32}$$

**(b)** Show that the gauge field has the following transformation property:

$$A'_\mu = UA_\mu U^\dagger - (\partial_\mu U)U^\dagger. \tag{8.33}$$

**Solution to Problem 8.2**

**(a)** From the definition of

$$\begin{aligned}\tilde{\psi}(x+dx) &= \psi(x+dx) + A_\mu(x)\psi(x)\,dx^\mu \\ &= \psi(x) + \partial_\mu\psi(x)\,dx^\mu + A_\mu(x)\psi(x)\,dx^\mu\end{aligned} \tag{8.34}$$

we see that the field difference is

$$\begin{aligned}\tilde{\psi}(x+dx) - \psi(x) &= \partial_\mu\psi(x)\,dx^\mu + A_\mu(x)\psi(x)\,dx^\mu \\ &= (\partial_\mu + A_\mu)\psi\,dx^\mu.\end{aligned} \tag{8.35}$$

Then the covariant derivative, being directly related to the left-hand side of this equation through the definition (8.31), $D_\mu\psi dx^\mu$, can be expressed directly in terms of the gauge field as given on the right-hand side:

$$D_\mu\psi = (\partial_\mu + A_\mu)\psi. \tag{8.36}$$

**(b)** By construction, the combination $\tilde{\psi}$ of $\psi$ and gauge field $A_\mu$ as given in (8.29) has a simple local transformation:

$$\tilde{\psi}'(x+dx) = U(x)\tilde{\psi}(x+dx). \tag{8.37}$$

From this we can discover the required transformation property of the gauge field by substituting in the relation (8.29) on both sides of the equation:

$$\begin{aligned} \text{LHS} &= \psi'(x+dx) + A'_\mu\, dx^\mu \psi' = U(x+dx)\psi(x+dx) + A'_\mu\, dx^\mu U(x)\psi \\ &= U(x)\psi(x) + (\partial_\mu U\psi + U\partial_\mu\psi)\, dx^\mu + A'_\mu\, dx^\mu U(x)\psi \end{aligned} \tag{8.38}$$

$$\text{RHS} = U(x)\left[\psi(x) + \partial_\mu\psi\, dx^\mu + A_\mu\, dx^\mu\psi\right]. \tag{8.39}$$

Equating these two expressions, we get

$$(\partial_\mu U)\psi + A'_\mu U\psi = UA_\mu\psi. \tag{8.40}$$

Eliminating $\psi$ which is arbitrary, we have

$$A'_\mu = UA_\mu U^\dagger - (\partial_\mu U)U^\dagger. \tag{8.41}$$

**Remark 1.** Because the transformation is position dependent, we have the term $(\partial_\mu U)U^\dagger$ in the transformation of the gauge field. This means that $A_\mu$ does not transform homogeneously as the ordinary field, being in some definite representation of the symmetry group. But this extra term is of the same form but of opposite sign so as to cancel the corresponding term in the transformation of the ordinary derivative. In this way the gauge field is just the compensating field needed to enforce the invariance of the theory under transformations that differ from point to point.

**Remark 2.** In this derivation of the transformation property of the gauge field we have emphasized the geometric aspect of a local transformation as discussed in CL-Section 8.2. In Section 8.1 we have already provided a derivation by explicitly using the covariant nature of the covariant derivative: the covariant derivative is defined to transform in the same way as $\psi(x)$, i.e.

$$D_\mu\psi(x) \to [D_\mu\psi(x)]' = U(x)[D_\mu\psi(x)]. \tag{8.42}$$

The gauge field $A_\mu$, having been introduced as the difference between the covariant and ordinary derivatives

$$D_\mu\psi(x) = \partial_\mu\psi(x) + A_\mu(x)\psi(x), \tag{8.43}$$

can easily be shown to have the transformation as given in (8.41).

## 8.3 General relativity as a gauge theory

In general relativity, one studies the *general coordinate transformation*

$$dx^\mu \to dx'^\mu = U^\mu_{\ \nu}(x)\,dx^\nu, \qquad \partial_\mu \to \partial'_\mu = [U^{-1}(x)]^\nu_{\ \mu}\partial_\nu \tag{8.44}$$

which can be viewed as a local transformation with

$$U^\mu_{\ \nu}(x) = \frac{\partial x'^\mu}{\partial x^\nu}, \qquad \left[U^{-1}(x)\right]^\nu_{\ \mu} = \frac{\partial x^\nu}{\partial x'^\mu}. \tag{8.45}$$

Following the same procedure as suggested in Problem 8.2, we can choose to view general relativity also as a gauge theory. Just as in eqn (8.44), a vector field $\xi^\mu(x)$ and a (mixed) tensor field $T^\mu_{\ \nu}(x)$ have the transformation properties of

$$\xi'^\mu(x) = U^\mu_{\ \nu}(x)\xi^\nu(x), \tag{8.46}$$

$$T'^\mu_{\ \lambda}(x) = U^\mu_{\ \nu}(x)[U^{-1}(x)]^\rho_{\ \lambda}T^\nu_{\ \rho}(x). \tag{8.47}$$

Consider the differentiation of a field having a non-trivial transformation property—the simplest case would be the vector field $\xi^\mu(x)$. Clearly, the ordinary derivative $\partial_\lambda\xi^\mu$ does not transform homogeneously, as in eqn (8.47), because the transformation is local, $\partial_\lambda U^\mu_{\ \nu} \neq 0$. Or, stated more geometrically, this is because $\xi^\mu(x+dx)$ transforms differently from $\xi^\mu(x)$. As in Problem 8.2 we can introduce, as in eqn (8.29), a modified vector field

$$\tilde{\xi}^\mu(x+dx) = \xi^\mu(x+dx) - \Gamma^\mu_{\alpha\beta}\xi^\alpha(x)\,dx^\beta \tag{8.48}$$

which transforms identically as $\xi^\mu(x)$:

$$\tilde{\xi}'^\mu(x+dx) = U^\mu_{\ \nu}(x)\tilde{\xi}^\nu(x+dx). \tag{8.49}$$

A comparison of eqns (8.29) and (8.48) shows that the compensating field $\Gamma^\mu_{\alpha\beta}$ plays the same role as the gauge field $A^\mu$. The $\Gamma^\mu_{\alpha\beta}$ field is called a *connection* in geometry or the *Christoffel symbol*.

**(a)** Show that if we define the covariant derivative by

$$\tilde{\xi}^\alpha(x+dx) - \xi^\alpha(x) \equiv D_\mu\xi^\alpha(x)\,dx^\mu + O((dx)^2), \tag{8.50}$$

then

$$D_\lambda\xi^\mu = \partial_\lambda\xi^\mu - \Gamma^\mu_{\lambda\nu}\xi^\nu. \tag{8.51}$$

**(b)** Show that the connection has the following transformation property:

$$\Gamma'^\mu_{\nu\lambda} = U^\mu_{\alpha}(U^{-1})^\beta_{\ \nu}(U^{-1})^\gamma_{\ \lambda}\Gamma^\alpha_{\beta\gamma} + (U^{-1})^\beta_{\ \nu}(U^{-1})^\gamma_{\ \lambda}\left(\partial_\beta U^\mu_{\ \gamma}\right). \tag{8.52}$$

Namely, besides the usual homogeneous term (first one on the right-hand side), there is an inhomogeneous term (the second one).

**(c)** Show that the Lagrangian for the vector field $\xi^\mu(x)$ given by

$$\mathcal{L} = \tfrac{1}{2}\left(D_\mu\xi^\alpha\right)\left(D_\nu\xi^\beta\right)g^{\mu\nu}g_{\alpha\beta} \tag{8.53}$$

is invariant under the general coordinate transformation. $g_{\alpha\beta}(x)$ is the position-dependent metric tensor.

**Solution to Problem 8.3**

**(a)** From the definition of

$$\begin{aligned}\tilde{\xi}^{\alpha}(x+dx) &= \xi^{\alpha}(x+dx) - \Gamma^{\alpha}_{\nu\rho}\, dx^{\nu}\xi^{\rho} \\ &= \xi^{\alpha}(x) + \partial_{\mu}\xi^{\alpha}(x)\, dx^{\mu} - \Gamma^{\alpha}_{\nu\rho}(x)\xi^{\rho}(x)\, dx^{\nu} \end{aligned} \tag{8.54}$$

we see that the field difference is

$$\begin{aligned}\tilde{\xi}^{\alpha}(x+dx) - \xi^{\alpha}(x) &= \partial_{\mu}\xi^{\alpha}(x)\, dx^{\mu} - \Gamma^{\alpha}_{\nu\rho}(x)\xi^{\rho}(x)\, dx^{\nu} \\ &= \left(\partial_{\mu}\xi^{\alpha} - \Gamma^{\alpha}_{\mu\rho}\xi^{\rho}\right) dx^{\mu}. \end{aligned} \tag{8.55}$$

Then the covariant derivative, being directly related to the left-hand side of this equation through the definition (8.50), $D_{\mu}\xi^{\alpha}\, dx^{\mu}$, can be expressed in terms of the gauge field in the right-hand side:

$$D_{\mu}\xi^{\alpha} = \partial_{\mu}\xi^{\alpha} - \Gamma^{\alpha}_{\mu\rho}\xi^{\rho}. \tag{8.56}$$

**(b)** Under the general coordinate (gauge) transformation, we have

$$\tilde{\xi}'^{\mu}(x+dx) = U^{\mu}_{\ \nu}(x)\tilde{\xi}^{\nu}(x+dx). \tag{8.57}$$

We can extract the transformation property of the connection field by writing out the components on both sides of this equation:

$$\begin{aligned}\text{LHS} &= \xi'^{\mu}(x+dx) - \Gamma'^{\mu}_{\lambda\rho}\, dx'^{\lambda}\xi'^{\rho} \\ &= U^{\mu}_{\ \nu}(x+dx)\xi^{\nu}(x+dx) - \Gamma'^{\mu}_{\lambda\rho}U^{\lambda}_{\ \alpha}\, dx^{\alpha}U^{\rho}_{\ \beta}\xi^{\beta} \\ &= U^{\mu}_{\ \nu}(x)\xi^{\nu}(x) + U^{\mu}_{\ \nu}\, dx^{\beta}\partial_{\beta}\xi^{\nu} + \partial_{\alpha}U^{\mu}_{\ \nu}\, dx^{\alpha}\xi^{\nu} \\ &\quad - \Gamma'^{\mu}_{\lambda\rho}U^{\lambda}_{\ \alpha}\, dx^{\alpha}U^{\rho}_{\ \beta}\xi^{\beta}. \end{aligned} \tag{8.58}$$

$$\text{RHS} = U^{\mu}_{\ \nu}(x)\left[\xi^{\nu}(x) + dx^{\beta}\partial_{\beta}\xi^{\nu} - \Gamma^{\nu}_{\alpha\beta}\, dx^{\alpha}\xi^{\beta}\right]. \tag{8.59}$$

Equating these two expressions, we get

$$\partial_{\alpha}U^{\mu}_{\ \beta}\, dx^{\alpha}\xi^{\beta} - \Gamma'^{\mu}_{\lambda\rho}U^{\lambda}_{\ \alpha}\, dx^{\alpha}U^{\rho}_{\ \beta}\xi^{\beta} = -U^{\mu}_{\ \nu}\Gamma^{\nu}_{\alpha\beta}\, dx^{\alpha}\xi^{\beta} \tag{8.60}$$

Since $dx^{\alpha}$ and $\xi^{\beta}$ are arbitrary, we have

$$\partial_{\alpha}U^{\mu}_{\ \beta} - \Gamma'^{\mu}_{\lambda\rho}U^{\lambda}_{\ \alpha}U^{\rho}_{\ \beta} = -U^{\mu}_{\ \nu}\Gamma^{\nu}_{\alpha\beta} \tag{8.61}$$

or, written in the form as shown in (8.52):

$$\Gamma'^{\mu}_{\lambda\rho} = (U^{-1})^{\beta}_{\ \lambda}(U^{-1})^{\gamma}_{\ \rho}U^{\mu}_{\ \alpha}\Gamma^{\alpha}_{\beta\gamma} + (U^{-1})^{\beta}_{\ \lambda}(U^{-1})^{\gamma}_{\ \rho}\left(\partial_{\beta}U^{\mu}_{\ \gamma}\right). \tag{8.62}$$

**Remark.** We have derived the transformation rule of the Christoffel symbols by starting with the combination (8.48) having a simple transformation property as in eqn (8.49). This approach emphasizes the geometric aspects of gauge transformation. Alternatively, we could have started with the definition a covariant derivative $D_\lambda \xi^\mu$ as in (8.51) which has the definite transformation property as a mixed tensor [cf. in eqn (8.47)]:

$$(D_\lambda \xi^\mu)' = U^\mu_\nu(x)[U^{-1}(x)]^\rho_\lambda \left(D_\rho \xi^\nu\right). \tag{8.63}$$

The same transformations of the Christoffel symbol can be extracted.

**(c)** As the covariant derivative $D_\alpha \xi^\mu$ and the metric $g_{\mu\nu}$ transform as tensors under the general coordinate transformation, it is easy to see that the Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\left(D_\mu \xi^\alpha\right)\left(D_\nu \xi^\beta\right) g^{\mu\nu} g_{\alpha\beta} \tag{8.64}$$

is invariant because all the tensor indices are contracted.

**Remark.** Just as the field tensor $F_{\alpha\beta}$ in gauge theory can be defined through the commutator of covariant derivatives,

$$(D_\alpha D_\beta - D_\beta D_\alpha)\psi \equiv F_{\alpha\beta}\psi, \tag{8.65}$$

the corresponding field tensor in general relativity can be defined in a similar way:

$$(D_\alpha D_\beta - D_\beta D_\alpha)B^\mu \equiv R^\mu_{\alpha\beta\nu} B^\nu. \tag{8.66}$$

The field tensor $R^\gamma_{\alpha\beta\delta}$ where

$$R^\mu_{\alpha\beta\nu} = \partial_\alpha \Gamma^\mu_{\gamma\beta} - \partial_\beta \Gamma^\mu_{\gamma\alpha} + \Gamma^\mu_{\rho\beta}\Gamma^\rho_{\gamma\alpha} - \Gamma^\mu_{\rho\alpha}\Gamma^\rho_{\gamma\beta} \tag{8.67}$$

is called the *Riemann curvature tensor*. From this, an invariant action of the gravitational field can be constructed.

## 8.4 O($n$) gauge theory

Consider two sets of scalar fields, $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, which transform as vector representations under the O($n$) group.

**(a)** Show that under infinitesimal O($n$) transformations we have

$$(\phi_\alpha)_i \to (\phi_\alpha)_i + \varepsilon_{ij}(\phi_\alpha)_j \qquad \text{where } \alpha = 1, 2 \tag{8.68}$$

and $\varepsilon_{ij} = -\varepsilon_{ji}$ are the parameters which characterize the infinitesimal O($n$) transformations.

**(b)** Construct the covariant derivative for $\boldsymbol{\phi}_1$.

**Solution to Problem 8.4**

**(a)** O$(n)$ transformations are characterized by $n \times n$ orthogonal matrices $O_{ij}$,

$$O_{ij} O_{ik} = \delta_{jk}. \tag{8.69}$$

For infinitesimal transformations, we write

$$O_{ij} = \delta_{ij} + \varepsilon_{ij}, \quad \text{with } \varepsilon_{ij} \ll 1 \tag{8.70}$$

and the orthogonality relation implies

$$\delta_{jk} + (\varepsilon_{jk} + \varepsilon_{kj}) = \delta_{jk} \quad \Rightarrow \quad \varepsilon_{ij} = -\varepsilon_{ji}. \tag{8.71}$$

Thus infinitesimal O$(n)$ transformations are characterized by $\frac{1}{2}n(n-1)$ parameters. In general, O$(n)$ vectors transform by $n \times n$ orthogonal matrices,

$$\phi_i \to \phi'_i = O_{ij}\phi_j. \tag{8.72}$$

For the infinitesimal transformations, we have

$$\phi'_i = \phi_i + \varepsilon_{ij}\phi_j. \tag{8.73}$$

**(b)** For the covariant derivative we need the adjoint representation of O $(n)$. It is not hard to see that they are just the second-rank antisymmetric tensors,

$$\phi_{ij} \to \phi'_{ij} = \phi_{ij} + (\varepsilon_{ik}\phi_{kj} + \varepsilon_{jk}\phi_{ik}) \quad \text{with } \phi_{ij} = -\phi_{ji} \tag{8.74}$$

This gives the global transformation law for the gauge bosons $W_{\mu ij}$. Write the covariant derivative of $\boldsymbol{\phi}$ as

$$D_\mu \phi_i = \partial_\mu \phi_i + g W_{\mu ik}\phi_k \quad \text{with } W_{\mu ik} = -W_{\mu ki}. \tag{8.75}$$

Then by requiring the covariant derivative of $\boldsymbol{\phi}$ to transform in the same way as $\phi_i$

$$(D_\mu \phi_i)' = D_\mu \phi_i + \varepsilon_{ij} D_\mu \phi_j \tag{8.76}$$

which can be written out as

$$\begin{aligned} \text{LHS} &= \partial_\mu \phi'_i + g W'_{\mu ik}\phi'_k = \partial_\mu(\phi_i + \varepsilon_{ij}\phi_j) + g W'_{\mu ik}(\phi_k + \varepsilon_{kj}\phi_j) \\ &= \partial_\mu \phi_i + (\partial_\mu \varepsilon_{ij})\phi_j + \varepsilon_{ij}\partial_\mu \phi_j + g W'_{\mu ik}(\phi_k + \varepsilon_{kj}\phi_j) \end{aligned} \tag{8.77}$$

$$\text{RHS} = \partial_\mu \phi_i + g W_{\mu ik}\phi_k + \varepsilon_{ij}(\partial_\mu \phi_j + g W_{\mu jk}\phi_k). \tag{8.78}$$

Combining these two expressions and cancelling out $\phi_i$ we get

$$\partial_\mu \varepsilon_{ij} + g W'_{\mu ik}(\delta_{kj} + \varepsilon_{kj}) = g W_{\mu ij} + g\varepsilon_{ik} W_{\mu kj}. \tag{8.79}$$

Multiplying by $(\delta_{jl} + \varepsilon_{jl})$ and using the property $\varepsilon_{ij} = -\varepsilon_{ji}$, we get

$$\partial_\mu \varepsilon_{il} + g W'_{\mu il} = g W_{\mu il} + g W_{\mu ij}\varepsilon_{jl} + g\varepsilon_{ik} W_{\mu kl} \tag{8.80}$$

or

$$W'_{\mu il} = W_{\mu il} + W_{\mu ij}\varepsilon_{jl} + \varepsilon_{ik} W_{\mu kl} - \frac{1}{g}(\partial_\mu \varepsilon_{il}). \tag{8.81}$$

This is the transformation law for the gauge bosons.

## 8.5 Broken generators and Goldstone bosons

Let $\phi_i$ be the scalar fields in the vector representation of the SU$(n)$ group.

**(a)** Write down the $SU(n)$ invariant scalar potential for $\phi_i$.

**(b)** Work out the possible pattern for the spontaneous symmetry breaking for $\phi_i$. How many Goldstone bosons are there in this case ?

**(c)** Discuss the possible spontaneous symmetry breaking pattern for the case where there are two such scalar fields $\phi_{1i}$ and $\phi_{2j}$.

### Solution to Problem 8.5

**(a)** As we have seen in Problem 8.1, the vector $\phi_i$ and its complex conjugate $\phi^i$ have the following transformation laws,

$$\phi_i \to \phi'_i = \phi_i + i\varepsilon_i{}^l\phi_l, \qquad \phi^i \to \phi'^i = \phi^i - i\varepsilon^i{}_k\phi^k. \tag{8.82}$$

Thus the SU$(n)$ invariant combination is of the form

$$\begin{aligned}\phi_i\phi^i \to \phi'_i\phi'^i &= \left(\phi_i + i\varepsilon_i{}^l\phi_l\right)\left(\phi^i - i\varepsilon^i{}_k\phi^k\right)\\ &= \phi_i\phi^i + i\varepsilon_i{}^l\phi_l\phi^i - i\varepsilon^i{}_k\phi^k\phi_i = \phi_i\phi^i\end{aligned} \tag{8.83}$$

which is just the scalar product in the $n$-dimensional complex vector space. The SU$(n)$ invariant scalar potential can depend on this combination $\phi_i\phi^i$,

$$V(\phi) = -\mu^2\phi_i\phi^i + \frac{\lambda}{2}\left(\phi_i\phi^i\right)^2. \tag{8.84}$$

**(b)** Let $\phi_i\phi^i = \phi_i\phi_i^* = \rho^2$. We can write the scalar potential as

$$V(\phi) = -\mu^2\rho^2 + \frac{\lambda}{2}\rho^4. \tag{8.85}$$

Then

$$\frac{\partial V}{\partial\rho} = (-\mu^2 + \lambda\rho^2)2\rho = 0 \quad\Rightarrow\quad \rho = \sqrt{\frac{\mu^2}{\lambda}} \equiv v \tag{8.86}$$

or

$$\phi_i\phi_i^* = \rho^2 = \frac{\mu^2}{\lambda}. \tag{8.87}$$

Without any loss of generality, we can choose

$$\langle\phi_i\rangle_0 = \delta_{in}v. \tag{8.88}$$

Clearly, the symmetry-breaking pattern is

$$\mathrm{SU}(n) \to \mathrm{SU}(n-1). \tag{8.89}$$

To get the Goldstone bosons, we write the fields as

$$\phi_i = \phi'_i + \delta_{in}v$$

so that $\phi_i'$ have zero VEV. It is easy to see that

$$\left(\phi^i\phi_i\right) = \left(\phi_i' + \delta_{in}v\right)\left(\phi^{i\prime} + \delta^{in}v\right) = \phi'^i\phi_i' + \left(\phi_n + \phi_n^*\right)v + v^2 \tag{8.90}$$

$$\begin{aligned}\left(\phi^i\phi_i\right)^2 &= \left[\phi'^i\phi_i' + \left(\phi_n + \phi_n^*\right)v + v^2\right]^2 \\ &= v^2\left(\phi_n + \phi_n^*\right)^2 + 2v^2\left(\phi'^i\phi_i'\right) + \cdots\end{aligned} \tag{8.91}$$

and the quadratic terms in $V(\phi)$ is

$$\begin{aligned}V_2(\phi) &= -\mu^2\left(\phi'^i\phi_i'\right) + \frac{\lambda}{2}\left[v^2\left(\phi_n + \phi_n^*\right)^2 + 2v^2\left(\phi'^i\phi_i'\right)\right] \\ &= \frac{\mu^2}{2}\left(\phi_n + \phi_n^*\right)^2.\end{aligned} \tag{8.92}$$

This means that $\phi_1, \phi_2, \ldots, \phi_{n-1}$ and $\mathrm{Im}\,\phi_n$ are massless Goldstone bosons. Since each of $\phi_1, \phi_2, \ldots, \phi_{n-1}$is a complex field and has two degrees of freedoms, the total number of Goldstone bosons is $(2n-1)$. This is precisely the number of broken generators, $(n^2-1) - [(n-1)^2 - 1]$, in the symmetry-breaking pattern $\mathrm{SU}(n) \to \mathrm{SU}(n-1)$.

**(c)** For the case of two vectors, it is easy to see that the $\mathrm{SU}(n)$ invariant combinations are of the form

$$\phi_{1i}\phi^{1i}, \quad \phi_{2i}\phi^{2i}, \quad \phi_{1i}\phi^{2i}, \quad \phi_{2i}\phi^{1i}. \tag{8.93}$$

We can parametrize them as

$$\phi_{1i}\phi^{1i} = \rho_1^2, \quad \phi_{2i}\phi^{2i} = \rho_2^2, \quad \phi_{1i}\phi^{2i} = \rho_1\rho_2 z, \quad \phi_{2i}\phi^{1i} = \rho_1\rho_2 z^*. \tag{8.94}$$

Then the scalar potential $V(\phi_1, \phi_2)$ can depend on $\rho_1$, $\rho_2$, and $z$. The minimization of $V(\phi_1, \phi_2)$ will fix the value of these three variables. Without loss of generality, we can choose the VEV to be

$$\langle\phi_1\rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \rho_1 \end{pmatrix}, \qquad \langle\phi_2\rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \rho_2(1-|z|^2)^{1/2} \\ \rho_2 z^* \end{pmatrix}. \tag{8.95}$$

The symmetry-breaking pattern is

$$\mathrm{SU}(n) \to \mathrm{SU}(n-2).$$

Note that just like vectors in $\mathrm{SO}(n)$, we can generalize this to the $k$ vectors to get the symmetry breaking

$$\mathrm{SU}(n) \to \mathrm{SU}(n-k).$$

## 8.6 Symmetry breaking by an adjoint scalar

Suppose ${\phi_i}^j$ are scalar fields in the adjoint representation of an SU $(n)$ group.

**(a)** Write down the scalar potential for ${\phi_i}^j$.

**(b)** Work out the possible pattern for the spontaneous symmetry breaking for ${\phi_i}^j$.

**Solution to Problem 8.6**

**(a)** From the SU$(n)$ transformation properties for the adjoint representation,

$$ {\phi_i}^j \to {\phi_i'}^j = {\phi_i}^j + i{\varepsilon_i}^l{\phi_l}^j - i{\varepsilon_k}^j{\phi_i}^k. \tag{8.96} $$

Note that ${\phi_i}^i = 0$ and $\left({\phi_i}^j\right)^* = {\phi_j}^i$. We have the following quadratic and quartic invariants (obtained by contracting the tensor indices):

$$ \phi_i^j\phi_j^i = Tr(\phi^2), \quad \left(\phi_i^j\phi_j^i\right)^2 = [Tr(\phi^2)]^2, \quad \phi_i^j\phi_j^k\phi_k^l\phi_l^i = Tr(\phi^4) \tag{8.97} $$

where we have written ${\phi_i}^j$ as an $n \times n$ matrix. The scalar potential takes the form

$$ V(\phi) = -\mu^2 Tr(\phi^2) + \lambda_1[Tr(\phi^2)]^2 + \lambda_2 Tr(\phi^4). \tag{8.98} $$

For simplicity, we have imposed a discrete symmetry $\phi \to -\phi$ to remove the term of the form $Tr\phi^3$.

**(b)** It is clear that $\phi$ is a traceless hermitian matrix, which can be diagonalized by the SU$(n)$ transformation. Thus without any loss of generality, we can take $\phi$ to be real and diagonal,

$$ \phi = \begin{pmatrix} \phi_1 & & & & \\ & \phi_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \phi_n \end{pmatrix} \quad \text{with } \phi_1 + \phi_2 + \cdots + \phi_n = 0. \tag{8.99} $$

The scalar potential is then of the form

$$ V(\phi) = -\mu^2 \sum_i \phi_i^2 + \lambda_1 \left[\sum_i \phi_i^2\right]^2 + \lambda_2 \sum_i \phi_i^4. \tag{8.100} $$

Since not all $\phi_i$s are independent, we need to introduce the Lagrange multiplier to take into account the constraint $\phi_1 + \phi_2 + \cdots + \phi_n = 0$:

$$ V(\phi) = -\mu^2 \sum_i \phi_i^2 + \lambda_1 \left[\sum_i \phi_i^2\right]^2 + \lambda_2 \sum_i \phi_i^4 - \xi\left(\sum_i \phi_i\right) \tag{8.101} $$

and $\xi$ is determined at the end by the condition

$$ \phi_1 + \phi_2 + \cdots + \phi_n = 0. \tag{8.102} $$

For the minimum of $V(\phi)$, we have

$$\frac{\partial V}{\partial \phi_i} = -2\mu^2\phi_i + 4\lambda_1\left(\sum_j \phi_j^2\right)\phi_i + 4\lambda_2\phi_i^3 - \xi = 0 \tag{8.103}$$

which means that each $\phi_i$ is a solution of the cubic equation

$$-2\mu^2 x + 4\lambda_1 a x + 4\lambda_2 x^3 - \xi = 0, \quad \text{with } a = \sum_i \phi_i^2. \tag{8.104}$$

Since the cubic equation can have at most three different roots, the most general form for $\phi_i$ is

$$\phi = \begin{pmatrix} \phi_1 & & & & \\ & \phi_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \phi_n \end{pmatrix} = \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_2 & & & \\ & & & \ddots & & \\ & & & & x_3 & \\ & & & & & \ddots \end{pmatrix} \tag{8.105}$$

with

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = 0, \quad n_1 + n_2 + n_3 = n. \tag{8.106}$$

Thus the general pattern for the symmetry breaking is

$$\mathrm{SU}(n) \to \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{SU}(n_3) \quad \text{with } n_1 + n_2 + n_3 = n. \tag{8.107}$$

In other words, the SU($n$) group with scalars in the adjoint representation can break at most into products of three SU($n_i$) groups. A more detailed calculation (Li 1974) shows the following symmetry-breaking patterns:

$$\mathrm{SU}(n) \to \mathrm{SU}(n_1) \times \mathrm{SU}(n - n_1) \times U(1) \quad \text{for } \lambda_2 > 0 \tag{8.108}$$

with

$$n_1 = \frac{n}{2}, \quad n \text{ even}, \qquad n_1 = \frac{n+1}{2}, \quad n \text{ odd}, \tag{8.109}$$

and for $\lambda_2 < 0$,

$$\mathrm{SU}(n) \to \mathrm{SU}(n-1) \times \mathrm{U}(1).$$

**Remark.** For the special case where $\lambda_1 = 0$, and the constraint $Tr\phi = 0$ is absent (so that $\xi = 0$), we have

$$V(\phi) = \sum_i \left(-\mu^2\phi_i^2 + \lambda_2\phi_i^4\right) = \sum_i f(\phi_i). \tag{8.110}$$

This means that $\phi_i$s decouple from each other. Thus each $f(\phi_i)$ is minimized independently, and it is easy to see that all $\phi_i$s should take the same value at the minimum of $f(\phi)$. Then the $VEV$ is of the form

$$\langle\phi\rangle_0 = v\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}. \tag{8.111}$$

This situation is realized in the spontaneous symmetry breaking of QCD in the large $N_c$ approximation (see Coleman and Witten 1980).

## 8.7 Symmetry breaking and the coset space

Suppose the scalar potential $V(\phi)$ is invariant under the symmetry group $G$.

**(a)** Show that if $\phi = \phi_1 \neq 0$ is a minimum of $V(\phi)$, then other $\phi_i$s, which are related to $\phi_i$ by symmetry transformations of $G$, also minimize $V(\phi)$. (Vacuum is necessarily degenerate.) Show that the symmetry operations that relate these $\phi_i$s form a subgroup, call it $H$, of $G$. The pattern of symmetry breaking is then $G \to H$.

**(b)** Because of the unbroken symmetry $H$, the minimum of $V(\phi)$ is a degenerate one, i.e. there is more than one value of $\phi$ which minimizes $V(\phi)$. Denote by $M(\phi_0)$ the set of $\phi$s which minimize $V(\phi)$. Show that for a given pattern of symmetry breaking, $G \to H$, $M(\phi_0)$ can be identified with the coset space $G/H$.

**(c)** For the case $G = \mathrm{SO}(n)$ and the scalar fields in the vector representation, the coset space is $\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}$, which is the surface of a sphere in $n$-dimensional real space.

### Solution to Problem 8.7

**(a)** $V(\phi)$ is invariant under $G$ means that

$$V(\phi) = V(g\phi) \qquad \forall\, g \in G$$

where $g\phi$ is obtained from $\phi$ by the transformation $\phi \to g\phi$ with $g \in G$. Then if $\phi_1 \neq 0$ is a minimum of $V(\phi)$, $g\phi_1$ is also a minimum of $V(\phi)$. It is clear that those group elements which leave $\phi_1$ invariant form a subgroup, call it $H$. This can be seen as follows. If $h_1\phi_1 = \phi_1$, $h_2\phi_1 = \phi_1$, then $h_1h_2\phi_1 = \phi_1$ and $h_1^{-1}\phi_1 = \phi_1$. Thus if $h_1, h_2 \in H$, then $(h_1h_2) \in H$ and $h_1^{-1}, h_2^{-1} \in H$. In other words, $H$ is a subgroup. The pattern for the symmetry breaking is then $G \to H$.

**(b)** Recall from group theory that the coset space $G/H$ is made up of (left) cosets of the form $g_iH$, i.e. the collection of elements obtained by left-multiplying $g_i \notin H$ with the whole subgroup $H$. The cosets obtained this way have the property that either they are completely different (no elements in common) or they are identical. In particular, if $g_i \notin H$ and $g_j \notin H$, but $g_ig_j^{-1} \in H$, then $g_iH = g_jH$, i.e. two cosets are identical. But if $g_ig_j^{-1} \notin H$, then $g_iH$ and $g_jH$ have no elements in common.

Now let us look at the set $M(\phi_0)$. Clearly, for each $\phi_i \in M(\phi_0)$, we have $H\phi_i = \phi_i$. Suppose we choose an arbitrary $\phi_1 \in M(\phi_0)$. Then by the action of the coset $g_i H$ we have ($g_i \notin H$)

$$g_i H\phi_1 = g_i\phi_1 \neq \phi_1. \tag{8.112}$$

This means that the action of the coset $g_i H$ on $\phi_1$ will generate another element $\phi_2 = g_i H\phi_1$, which is different from $\phi_1$. Since the potential $V(\phi)$ is invariant under the group $G$, the new element $\phi_2$ must also be in $M(\phi_0)$. Furthermore,

$$g_i H\phi_1 \neq g_j H\phi_1 \qquad \text{if } g_i g_j^{-1} \notin H. \tag{8.113}$$

Namely, different cosets generate different $\phi_i$s in $M(\phi_0)$. Thus if we identify $\phi_1$ with $H$ and the image $g_i\phi_1$ with the coset $g_i H$, we have a one-to-one mapping of the $\phi$s in $M(\phi_0)$ with the coset $G/H$. This mapping is onto if $M(\phi_0)$ is transitive, i.e. every element in $M(\phi_0)$ can be obtained from a given $\phi_1$ by the action of a group element in $G$.

**(c)** For vector representation in SO$(n)$, the scalar potential $V(\phi)$ depends only on

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} = \phi_1^2 + \cdots + \phi_n^2 \tag{8.114}$$

and it is minimized for a particular value of this combination

$$\phi_1^2 + \cdots + \phi_n^2 = v^2, \tag{8.115}$$

where $v$ is related to the parameters in $V(\phi)$. Thus

$$M(\phi_0) = \left\{\boldsymbol{\phi} \,|\, \boldsymbol{\phi} \cdot \boldsymbol{\phi} = v^2\right\} \tag{8.116}$$

and it is just the surface of a sphere in an $n$-dimensional real space, $S^{n-1}$. Therefore, we have the result

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1} \tag{8.117}$$

**Remark.** It is easy to generalize this to the unitary group to get

$$\mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n-1}. \tag{8.118}$$

## 8.8 Scalar potential and first-order phase transition

Consider the case of one hermitian scalar field $\phi$ with scalar potential

$$V_0(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{8.119}$$

We have shown in the CL-eqn (5.134) that $V_0(\phi)$ has a degenerate minimum at $\phi = \pm v$, with $v = (\mu^2/\lambda)^{1/2}$. Suppose we add a cubic term to $V_0(\phi)$

$$V_0'(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{2\xi}{3}\phi^3 + \frac{\lambda}{4}\phi^4. \tag{8.120}$$

Show that the degeneracy in the minimum of $V_0(\phi)$ is now removed. Find the true minimum of $V_0'(\phi)$. Also, show that, as a function of the parameter $\xi$, the VEV $\langle\phi\rangle_0$ changes discontinuously from $\langle\phi\rangle_0 = -v$ to $\langle\phi\rangle_0 = v$ as $\xi$ changes from positive to negative values going through 0.

**Solution to Problem 8.8**

The minimization condition leads to the equation

$$\frac{\partial V_0'}{\partial \phi} = 0 \quad \Rightarrow \quad \phi(-\mu^2 + 2\xi\phi + \lambda\phi^2) = 0. \tag{8.121}$$

The non-trivial solutions are

$$\phi = \phi_\pm = \frac{1}{\lambda}\left[-\xi \pm (\xi^2 + \lambda\mu^2)^{1/2}\right]. \tag{8.122}$$

The two minima no longer have the same value for the potential $V_0'(\phi)$. For small $\xi$, we have

$$\phi_\pm = \pm v - \frac{\xi}{\lambda} + O(\xi^2) = \phi_0 - \frac{\xi}{\lambda} \tag{8.123}$$

with $\phi_0 = v$ or $-v$, and

$$V_0'(\phi_\pm) = V_0(v) + \left(\frac{\mu^2}{\lambda} - 2\phi_0^2\right)\phi_0\xi = V_0(v) - \frac{\mu^2}{\lambda}\phi_0\xi. \tag{8.124}$$

Then for $\xi > 0$, $V_0'(\phi)$ has minimum at $\phi_-$, while the minimum is at $\phi_+$ for $\xi < 0$. This means that as $\xi$ varies from $\xi < 0$ to $\xi > 0$, $\phi$ changes from $-v$ to $v$ discontinuously. This is usually referred to as a first-order phase transition.

## 8.9 Superconductivity as a Higgs phenomenon

Consider the scalar QED with Higgs phenomena with the Lagrangian

$$\mathcal{L} = (D_\mu\phi^\dagger)(D^\mu\phi) + \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{8.125}$$

with

$$D_\mu\phi = (\partial_\mu - ieA_\mu)\phi, \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{8.126}$$

Consider the static case where $\partial^0\phi = \partial^0\mathbf{A} = 0$ and $A_0 = 0$.

**(a)** Show that the equation of motion for $\mathbf{A}$ is of the form

$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} \qquad \text{with} \quad \mathbf{J} = ie[\phi^\dagger(\boldsymbol{\nabla} - ie\mathbf{A})\phi - (\boldsymbol{\nabla} + ie\mathbf{A})\phi^\dagger\phi].$$

**(b)** Show that with spontaneous symmetry breaking, in the classical approximation $\phi = v = (\mu^2/\lambda)^{1/2}$, the current $\mathbf{J}$ is of the form

$$\mathbf{J} = e^2v^2\mathbf{A} \qquad \text{(the London equation)} \tag{8.127}$$

and thus

$$\nabla^2\mathbf{B} = e^2v^2\mathbf{B} \qquad \text{(the Meissner effect)}. \tag{8.128}$$

**(c)** The resistivity $\rho$ for the system is defined by

$$\mathbf{E} = \rho\mathbf{J}. \tag{8.129}$$

Show that, in this case of spontaneous symmetry breaking, $\rho = 0$, and we have superconductivity.

**Solution to Problem 8.9**

**(a)** In the static limit, Maxwell's equations are of the form

$$-\partial_\mu F^{\mu\nu} = J^\nu \quad \Rightarrow \quad \partial_i F^{ij} = -j^j \quad \text{or} \quad \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} \tag{8.130}$$

where

$$\begin{aligned} J_\mu &= -\frac{\partial \mathcal{L}}{\partial A_\mu} = ie\left[\left(D_\mu \phi^\dagger\right)\phi - \phi^\dagger (D_\mu \phi)\right] \\ &= ie\left[(\partial_\mu + ieA_\mu)\phi^\dagger \phi - \phi^\dagger(\partial_\mu - ieA_\mu)\phi\right]. \end{aligned} \tag{8.131}$$

Thus

$$\mathbf{J} = ie\left[\phi^\dagger(\boldsymbol{\nabla} - ie\mathbf{A})\phi - (\boldsymbol{\nabla} + ie\mathbf{A})\phi^\dagger\phi\right]. \tag{8.132}$$

**(b)** Spontaneous symmetry breaking gives $\phi = v = (\mu^2/\lambda)^{1/2}$, which in turn gives the London equation:

$$\mathbf{J} = e^2 v^2 \mathbf{A}. \tag{8.133}$$

From Maxwell's equation, $\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J}$, we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{J} \qquad \text{or} \qquad \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -e^2 v^2 \boldsymbol{\nabla} \times \mathbf{A}$$

or

$$\nabla^2 \mathbf{B} = e^2 v^2 \mathbf{B} \tag{8.134}$$

where we have used $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. It is not difficult to see that this equation implies the Meissner effect because it implies a solution for the magnetic field of the form

$$B(\mathbf{x}) \simeq \exp\left(\frac{\mathbf{n} \cdot \mathbf{x}}{l}\right) \qquad \text{with} \quad l = \frac{1}{ev}. \tag{8.135}$$

This means that the magnetic fields decays in a distance of order of $l \sim (ev)^{-1}$.

**(c)** Since $\partial^0 \mathbf{A} = 0$ and $A^0 = 0$, we get $\mathbf{E} = 0$. On the other hand, we have $\mathbf{J} \neq 0$. This means that the resistivity must vanish (superconductivity) $\rho = 0$.

# 9 Quantum gauge theories

## 9.1 Propagator in the covariant $R_\xi$ gauge

The free field generating functional in the generalized covariant gauge is given in CL-eqn (9.82) as

$$W_A^0[J] = \int [d\mathbf{A}_\mu] \exp\left\{ i \int d^4x \left[ \frac{1}{2} A_\mu^a K_{ab}^{\mu\nu}(x) A_\nu^b + J_\mu^a A^{\mu a} \right] \right\} \tag{9.1}$$

with

$$K_{ab}^{\mu\nu}(x) = \delta_{ab} \left[ g^{\mu\nu} \partial^2 - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right]. \tag{9.2}$$

$\xi$ is an arbitrary constant. If we define the Green's function $G_{ab}^{\mu\nu}(x-y)$ by

$$\int d^4y\, K_{ab}^{\mu\nu}(x-y) G_{\nu\lambda}^{bc}(y-z) = g_\lambda^\mu \delta_a^c \delta^4(x-z) \tag{9.3}$$

where

$$K_{ab}^{\mu\nu}(x-y) = \delta^4(x-y) K_{ab}^{\mu\nu}(x), \tag{9.4}$$

show that

$$G_{ab}^{\mu\nu}(x-y) = \delta_{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \left[ -\left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) - \xi \frac{k^\mu k^\nu}{k^2} \right] \frac{1}{k^2} \tag{9.5}$$

is the propagator for the gauge field.

**Solution to Problem 9.1**

The definition of the Green's function can be written as

$$K_{ab}^{\mu\nu}(x) G_{\nu\lambda}^{bc}(x-z) = g_\lambda^\mu \delta_a^c \delta^4(x-z). \tag{9.6}$$

For the internal symmetry indices, we have $G_{\nu\lambda}^{bc} \propto \delta^{bc}$. We can thus define the Fourier transform as

$$G_{\nu\lambda}^{bc}(y-z) = \delta^{bc} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(y-z)} \tilde{g}_{\nu\lambda}(k). \tag{9.7}$$

We then get

$$\left[ -g^{\mu\nu} k^2 + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \right] \tilde{g}_{\nu\lambda}(k) = g_\lambda^\mu. \tag{9.8}$$

From general covariance, we can write

$$\tilde{g}_{\nu\lambda}(k) = a(k^2) g_{\nu\lambda} + b(k^2) k_\nu k_\lambda. \tag{9.9}$$

Then we get

$$a(k^2)\left[-g^\mu_\lambda k^2 + \left(1 - \frac{1}{\xi}\right) k^\mu k_\lambda\right] + b(k^2)\left(-\frac{1}{\xi}\right) k^2 k^\mu k_\lambda = g^\mu_\lambda. \tag{9.10}$$

By identifying independent tensors on both sides, we get

$$a(k^2) = -\frac{1}{k^2}, \qquad b(k^2) = \frac{1}{k^4}(1 - \xi). \tag{9.11}$$

Thus,

$$\tilde{g}_{\nu\lambda}(k) = \frac{1}{k^2}\left[-\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) - \xi \frac{k^\mu k^\nu}{k^2}\right]. \tag{9.12}$$

## 9.2 The propagator for a massive vector field

For a massive vector field $V_\mu$, the free Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2}{2} V^\mu V_\mu \qquad \text{with} \quad V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu.$$

Show that the propagator is given by

$$i\Delta_{\mu\nu}(k) = \frac{-i\left(g_{\mu\nu} - \left(k_\mu k_\nu / M^2\right)\right)}{k^2 - M^2 + i\varepsilon}. \tag{9.13}$$

**Solution to Problem 9.2**

We can write the generating functional as (see Problem 9.1),

$$\begin{aligned} W[J] &= \int [dV_\mu] \exp\left\{ i \int \left(\mathcal{L}_0 + J^\mu V_\mu\right) d^4x \right\} \\ &= \int [dV_\mu] \exp\left\{ i \int d^4x \left[\tfrac{1}{2} V_\mu(x) K^{\mu\nu}(x) V_\nu(x) + J^\mu V_\mu\right] \right\} \end{aligned} \tag{9.14}$$

where $K^{\mu\nu}$ can be calculated from $\mathcal{L}_0$ as follow:

$$\begin{aligned} \int d^4x\, \mathcal{L}_0 &= \int d^4x \left[ -\tfrac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) + \frac{M^2}{2} V^\mu V_\mu \right] \\ &= \int d^4x \left[ \tfrac{1}{2}(V_\nu \partial^2 V_\nu - V_\mu \partial^\mu \partial^\nu V_\nu) + \frac{M^2}{2} V^\mu V_\mu \right]. \end{aligned} \tag{9.15}$$

Comparing it to

$$\int d^4x\, \mathcal{L}_0 = \int d^4x\, \tfrac{1}{2} V_\mu(x) K^{\mu\nu}(x) V_\nu(x), \tag{9.16}$$

we can conclude that

$$K^{\mu\nu}(x) = (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + M^2 g^{\mu\nu}). \tag{9.17}$$

Define the Green's function $G_{\mu\nu}(x-y)$ by

$$K^{\mu\nu}(x)G_{\nu\lambda}(x-y) = g^\mu_\lambda \delta^4(x-y), \tag{9.18}$$

and introduce the Fourier transform

$$G_{\nu\lambda}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \tilde{g}_{\nu\lambda}(k) \tag{9.19}$$

to get

$$\left(-g^{\mu\nu}k^2 + k^\mu k^\nu + M^2 g^{\mu\nu}\right)\tilde{g}_{\nu\lambda}(k) = g^\mu_\lambda. \tag{9.20}$$

From the tensor structure $\tilde{g}_{\nu\lambda}(k) = a(k^2)g_{\nu\lambda} + b(k^2)k_\nu k_\lambda$, this equation can then be written as

$$\left[-g^{\mu\nu}(k^2 - M^2) + k^\mu k^\nu\right]\left[a(k^2)g_{\nu\lambda} + b(k^2)k_\nu k_\lambda\right] = g^\mu_\lambda,$$

which fixes the scalar function to be

$$a = -\frac{1}{k^2 - M^2}, \qquad b = \frac{1}{M^2(k^2 - M^2)}.$$

Thus we obtain the result

$$\tilde{g}_{\nu\lambda}(k) = \frac{1}{k^2 - M^2}\left[-g_{\nu\lambda} + \frac{k_\nu k_\lambda}{M^2}\right]. \tag{9.21}$$

## 9.3 Gauge boson propagator in the axial gauge

The axial gauge condition is given by

$$n^\mu A^a_\mu = 0 \tag{9.22}$$

where $n^\mu$ with $n^2 < 0$ is a space-like vector. Show that the gauge boson propagator in momentum space is of the form

$$\Delta^{\mu\nu}_{ab}(k) = \frac{\delta_{ab}}{k^2}\left[-g^{\mu\nu} + \frac{1}{k\cdot n}(n^\mu k^\nu + k^\mu n^\nu) + \frac{(\alpha k^2 - n^2)}{(n\cdot k)^2}k^\mu k^\nu\right] \tag{9.23}$$

where $\alpha$ is an arbitrary parameter in the gauge-fixing term.

**Solution to Problem 9.3**

Omitting the unimportant non-Abelian indices $a$, $b$, the free Lagrangian with gauge fixing term is given by

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2\alpha}(n\cdot A)^2. \tag{9.24}$$

Then

$$\begin{aligned} S_0 &= \int \mathcal{L}_0 d^4x = \int d^4x \left[\frac{1}{2}A^\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu + \frac{1}{2\alpha}A^\mu n_\mu n_\nu A^\nu\right] \\ &= \int d^4x \frac{1}{2}A^\mu K_{\mu\nu}A^\nu \end{aligned}$$

with

$$K_{\mu\nu} = (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) + \frac{1}{\alpha}n_\mu n_{\nu.}$$

The gauge boson propagator $\Delta_{\mu\nu}(x)$ is defined by

$$K_{\mu\nu}(x)\Delta^{\nu\lambda}(x-y) = g_\mu^\lambda \delta^4(x-y). \tag{9.25}$$

In momentum space, this corresponds to

$$K_{\mu\nu}(k)\Delta^{\nu\lambda}(k) = g_\mu^\lambda \qquad \text{with} \quad K_{\mu\nu} = (-g_{\mu\nu}k^2 + k_\mu k_\nu) + \frac{1}{\alpha}n_\mu n_\nu.$$

It is straightforward to solve for the coefficients $a$, $b$, $c$, and $d$ in the tensor decomposition of the propagator

$$\Delta^{\nu\lambda}(k) = ag^{\nu\lambda} + b(n_\mu k_\nu + k_\mu n_\nu) + ck_\mu k_\nu + dn_\mu n_\nu \tag{9.26}$$

and the results are

$$a = -\frac{1}{k^2}, \qquad b = \frac{1}{n\cdot k}\frac{1}{k^2}, \qquad c = -\frac{n^2-\alpha k^2}{k^2(n\cdot k)^2}, \qquad d = 0. \tag{9.27}$$

We obtain the stated form of

$$\Delta^{\nu\lambda}(k) = \frac{1}{k^2}\left[-g^{\nu\lambda} + \frac{1}{k\cdot n}(n_\mu k_\nu + k_\mu n_\nu) - \frac{(n^2-\alpha k^2)}{(n\cdot k)^2}k_\mu k_\nu\right]. \tag{9.28}$$

## 9.4 Gauge boson propagator in the Coulomb gauge

Calculate the gauge boson propagator in the Coulomb gauge:

$$\boldsymbol{\partial}\cdot\mathbf{A} = 0, \tag{9.29}$$

where we have ignored the internal symmetry index. To solve this problem we suggest rewriting this gauge condition as

$$\partial^\mu A_\mu - c_\mu\partial^\mu(c_\nu A^\nu) = 0 \quad \text{where} \quad c_\mu = (1,0,0,0).$$

**Solution to Problem 9.4**

The Lagrangian with the corresponding gauge-fixing term is given as

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}[\partial \cdot A - c \cdot \partial(c \cdot A)]^2. \tag{9.30}$$

Thus,

$$\int d^4x\,\mathcal{L}_0 = \int d^4x \Bigg\{ \frac{1}{2}\left[A^\mu(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu\right]$$
$$-\frac{1}{2\alpha}A^\mu\left[\partial_\mu\partial_\nu - A^\mu(c\cdot\partial)(c_\nu\partial_\mu + c_\mu\partial_\nu) + (c\cdot\partial)^2 c_\mu c_\nu\right]A^\nu\Bigg\}$$

and

$$K_{\mu\nu}(x) = (\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu) - \frac{1}{\alpha}\left[\partial_\mu\partial_\nu - (c\cdot\partial)(c_\nu\partial_\mu + c_\mu\partial_\nu) + (c\cdot\partial)^2 c_\mu c_\nu\right].$$

In momentum space, this gives

$$\begin{aligned} K_{\mu\nu}(k) &= \left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) \\ &\quad + \frac{1}{\alpha}\left[k_\mu k_\nu - (c\cdot k)\left(c_\nu k_\mu + c_\mu k_\nu\right) + (c\cdot k)^2 c_\mu c_\nu\right] \\ &= \left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) + \frac{1}{\alpha}\left[k_\mu - (c\cdot k)c_\mu\right]\left[k_\nu - (c\cdot k)c_\nu\right] \\ &= \left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) + \frac{1}{\alpha}n_\mu n_\nu \end{aligned} \tag{9.31}$$

where

$$n_\mu = k_\mu - (c\cdot k)c_\mu. \tag{9.32}$$

Since $K_{\mu\nu}(k)$ now has the same structure as in Problem 9.3, we can take over its result to write

$$\Delta_{\mu\nu}(k) = \frac{1}{k^2}\left[-g_{\mu\nu} + \frac{1}{k\cdot n}(n_\mu k_\nu + k_\mu n_\nu) - \frac{n^2 - \alpha k^2}{(n\cdot k)^2}k_\mu k_\nu\right]. \tag{9.33}$$

From

$$n^2 = [k - (c\cdot k)c]^2 = k^2 - (c\cdot k)^2, \qquad k\cdot n = k^2 - (c\cdot k)^2,$$

$$n_\mu k_\nu + k_\mu n_\nu = 2k_\mu k_\nu - (k\cdot c)(c_\mu k_\nu + k_\mu c_\nu),$$

we get

$$\begin{aligned} \Delta_{\mu\nu}(k) = -\frac{1}{k^2}\Bigg\{ & g_{\mu\nu} + \frac{c\cdot k}{k^2 - (k\cdot c)^2}(c_\mu k_\nu + k_\mu c_\nu) \\ & - \frac{k_\mu k_\nu}{k^2 - (c\cdot k)^2} - \frac{\alpha k^2 k_\mu k_\nu}{\left[k^2 - (c\cdot k)^2\right]^2}\Bigg\}. \end{aligned} \tag{9.34}$$

## 9.5 Gauge invariance of a scattering amplitude

In the non-Abelian gauge theory with fermions the Lagrangian is of the form

$$\mathcal{L} = -\tfrac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} + \bar{\psi}i\gamma^{\mu}D_{\mu}\psi - m\bar{\psi}\psi \tag{9.35}$$

with

$$D_{\mu}\psi = (\partial_{\mu} - igt^{a}A^{a}_{\mu})\psi. \tag{9.36}$$

Show that, to the lowest order in the gauge coupling $g$, the fermion scattering amplitude

$$\psi^{a} + \psi^{b} \to \psi^{c} + \psi^{d}$$

is the same in the covariant, the axial, and the Coulomb gauges.

**Solution to Problem 9.5**

A typical amplitude, to order $g^2$, has the structure

$$\begin{aligned}\mathcal{M}_{cd,ab} &= \left[\bar{u}(p_c)\gamma_{\mu}t^{i}_{cb}u(p_b)\right]\Delta^{\mu\nu}(k)\left[\bar{u}(p_d)\gamma_{\nu}t^{i}_{da}u(p_a)\right]\\ &= J_{\mu}(p_c, p_b)\Delta^{\mu\nu}(k)J_{\nu}(p_d, p_a)(t^{i}_{cb}t^{i}_{da})\end{aligned} \tag{9.37}$$

with

$$k = p_a - p_d = p_c - p_b, \qquad J_{\mu}(p_c, p_b) = \bar{u}(p_c)\gamma_{\mu}u(p_b).$$

It is easy to see that $J_{\mu}$ has the property

$$\begin{aligned}k^{\mu}J_{\mu}(p_c, p_b) &= (p_c - p_b)^{\mu}\bar{u}(p_c)\gamma_{\mu}u(p_b) = \bar{u}(p_c)(\not{p}_c - \not{p}_d)u(p_b)\\ &= (m_c - m_d)\bar{u}(p_c)u(p_b) = 0.\end{aligned} \tag{9.38}$$

Similarly,

$$k^{\nu}J_{\nu}(p_d, p_a) = (p_d - p_a)^{\nu}\bar{u}(p_d)\gamma_{\nu}u(p_a) = 0. \tag{9.39}$$

Vector boson propagators $\Delta^{\mu\nu}(k)$ in different gauges differ from each other by terms of the form $k_{\mu}k_{\nu}$ or $(k_{\mu}n_{\nu} + n_{\mu}k_{\nu})$. Since $k^{\mu}J_{\mu} = k^{\nu}J_{\nu} = 0$, these terms do not contribute to the amplitude $\mathcal{M}_{cd,ab}$. We get the same answer in all these gauges. Note that it is essential that the term proportional to $n_{\mu}n_{\nu}$ is absent in order to get the gauge independence results.

## 9.6 Ward identities in QED

The *generating functional* for QED in the covariant gauge is of the form (see Ryder 1985)

$$Z(J, \eta, \bar{\eta}) = N\int [dA_{\mu}][d\psi][d\bar{\psi}]\exp\left(i\int \mathcal{L}_{eff}\, d^4x\right) \tag{9.40}$$

where

$$\mathcal{L}_{eff} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}i(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi.$$

**(a)** Show that if we require the generating functional $Z(J, \eta, \bar{\eta})$ to be invariant under the (infinitesimal) gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda \qquad \text{and} \qquad \psi \to \psi - ie\Lambda\psi, \tag{9.41}$$

we get the relation called the *Ward–Takahashi identity*:

$$\left[\frac{i}{\alpha}\Box\partial^\mu \frac{\delta}{\delta J_\mu} - \partial^\mu J_\mu - e\left(\bar{\eta}\frac{\delta}{\delta\eta} - \eta\frac{\delta}{\delta\bar{\eta}}\right)\right] Z(J, \eta, \bar{\eta}) = 0. \tag{9.42}$$

**(b)** This identity (9.42) can be translated into an equation for the generating functional for the *connected Green's function* $W$ by $Z = e^{iW}$. Furthermore, we can write this in terms of the Legendre transform of $W[\eta, \bar{\eta}, J_\mu]$

$$\Gamma[\psi, \bar{\psi}, A_\mu] = W[\eta, \bar{\eta}, J_\mu] - \int d^4x(J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi) \tag{9.43}$$

where

$$\psi = \frac{\delta W}{\delta\bar{\eta}}, \qquad \bar{\psi} = \frac{\delta W}{\delta\eta}, \qquad A_\mu = \frac{\delta W}{\delta J_\mu}. \tag{9.44}$$

are usually called the classical fields.

**(c)** Show that by differentiating the result in part (b) with respect to $\bar{\psi}(x_1)$ and $\psi(y_1)$ and setting $\psi = \bar{\psi} = A_\mu = 0$ we can derive the familiar form of the Ward identity:

$$q^\mu \Gamma_\mu(p, q, p+q) = S_F^{-1}(p+q) - S_F^{-1}(p). \tag{9.45}$$

where the *vertex function* $\Gamma_\mu$ and the propagator $S_F^{-1}$ in momentum space are related to $\Gamma$ in (9.44) by

$$\int d^4x d^4x_1 d^4y_1 e^{i(p'x_1 - py_1 - qx)} \frac{\delta^3\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu} = ie(2\pi)^4\delta^4(p' - p - q)\Gamma_\mu(p, q, p') \tag{9.46}$$

and

$$\int d^4x_1\, d^4y_1\, e^{i(p'x_1 - py_1)} \frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} = ie(2\pi)^4\delta^4(p' - p)iS_F^{-1}(p). \tag{9.47}$$

**(d)** From part (b) show that

$$-\frac{1}{\alpha}\Box_x \partial_x^\mu \frac{\delta^2 W}{\delta J_\mu \delta J_\nu} = \partial_x^\mu g_{\mu\nu}\delta^4(x-y), \tag{9.48}$$

which in momentum space gives

$$\frac{i}{\alpha}k^2 k^\mu \tilde{G}_{\mu\nu}(k) = k_\nu \tag{9.49}$$

with

$$-\frac{\delta^2 W}{\delta J_\mu(x)\delta J_\nu(y)} = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}\tilde{G}_{\mu\nu}(k).$$

**Solution to Problem 9.6**

**(a)** Write

$$\mathcal{L}_{eff} = \mathcal{L} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi = \mathcal{L} + \mathcal{L}_1 \tag{9.50}$$

where $\mathcal{L}$ is the original Lagrangian without the gauge fixing and source terms. Recall that $\mathcal{L}$ is invariant under the gauge transformation. Thus the changes of $\mathcal{L}_{eff}$ under the gauge transformation all come from $\mathcal{L}_1$,

$$\begin{aligned}\delta\int \mathcal{L}_{eff}\, d^4x &= \delta\int \mathcal{L}_1\, d^4x \\ &= \int d^4x \left[-\frac{1}{\alpha}(\partial^\mu A_\mu)\Box\Lambda + J^\mu\partial_\mu\Lambda - ie\Lambda(-\bar{\psi}\eta + \bar{\eta}\psi)\right].\end{aligned} \tag{9.51}$$

The change in the generating functional due to the gauge transformation is then

$$\begin{aligned}\delta Z(J,\eta,\bar{\eta}) &= N\int [dA_\mu][d\psi][d\bar{\psi}]\exp\left(i\int \mathcal{L}_{eff}\, d^4x\right)\delta\int \mathcal{L}_{eff}\, d^4x \\ &= N\int [dA_\mu][d\psi][d\bar{\psi}]\exp\left(i\int \mathcal{L}_{eff}\, d^4x\right) \\ &\quad\times\left\{\int d^4x\left[-\frac{1}{\alpha}\Box(\partial^\mu A_\mu) - \partial_\mu J^\mu - ie(-\bar{\psi}\eta + \bar{\eta}\psi)\right]\Lambda(x)\right\}.\end{aligned}$$

Gauge invariance implies that $\delta Z = 0$. Since $\Lambda(x)$ in $\delta Z$ is arbitrary, its coefficient must vanish:

$$\left[\frac{i}{\alpha}\Box\left(\partial^\mu\frac{\delta}{\delta J^\mu}\right) - \partial_\mu J^\mu - e\left(-\eta\frac{\delta}{\delta\eta} + \bar{\eta}\frac{\delta}{\delta\bar{\eta}}\right)\right] Z(J,\eta,\bar{\eta}) = 0. \tag{9.52}$$

**(b)** Insert $Z = e^{iW}$ into the above equation and perform the differentiation to get

$$\left[\frac{1}{\alpha}\Box\left(\partial^{\mu}\frac{\delta W}{\delta J^{\mu}}\right) - \partial_{\mu}J^{\mu} - ie\left(-\eta\frac{\delta W}{\delta\eta} + \bar{\eta}\frac{\delta W}{\delta\bar{\eta}}\right)\right] = 0. \tag{9.53}$$

In terms of Legendre transform:

$$\Gamma[\psi, \bar{\psi}, A_{\mu}] = W[\eta, \bar{\eta}, J_{\mu}] - \int d^4x\,(J^{\mu}A_{\mu} + \bar{\psi}\eta + \bar{\eta}\psi) \tag{9.54}$$

where

$$\psi = \frac{\delta W}{\delta\bar{\eta}}, \qquad \bar{\psi} = \frac{\delta W}{\delta\eta}, \qquad A_{\mu} = \frac{\delta W}{\delta J_{\mu}}.$$

Then we have

$$-\eta = \frac{\delta\Gamma}{\delta\bar{\psi}}, \qquad -\bar{\eta} = \frac{\delta\Gamma}{\delta\psi}, \qquad -J_{\mu} = \frac{\delta\Gamma}{\delta A_{\mu}}$$

and we can write eqn (9.53) as

$$\left[\frac{1}{\alpha}\Box(\partial^{\mu}A_{\mu}) - \partial_{\mu}\frac{\delta\Gamma}{\delta A_{\mu}} - ie\left(-\psi\frac{\delta\Gamma}{\delta\psi} + \bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}}\right)\right] = 0. \tag{9.55}$$

**(c)** Differentiating eqn (9.55) with respect to $\bar{\psi}(x_1)$ and $\psi(y_1)$ and setting $\bar{\psi} = \psi = A_{\mu} = 0$, we get the relation

$$\begin{aligned}-\partial_x^{\mu}\frac{\delta^3\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^{\mu}} &= ie\delta(x - x_1)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} \\ &\quad - ie\delta(x - y_1)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)}.\end{aligned} \tag{9.56}$$

Multiplying by $\exp i(p'x - py_1 - qx)$ and integrating over $x$, $x_1$, and $y_1$, we get

$$\begin{aligned}&-\int d^4x\,d^4x_1\,d^4y_1\,e^{i(p'x_1 - py_1 - qx)}\partial_x^{\mu}\frac{\delta^3\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^{\mu}} \\ &= ie\int d^4x\,d^4x_1\,d^4y_1\,e^{i(p'x_1 - py_1 - qx)}\delta^4(x - x_1)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} \\ &\quad - ie\int d^4x\,d^4x_1\,d^4y_1\,e^{i(p'x_1 - py_1 - qx)}\delta^4(x - y_1)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)}.\end{aligned}$$

This gives

$$q^{\mu}\Gamma_{\mu}(p, q, p+q) = S_F^{-1}(p+q) - S_F^{-1}(p) \tag{9.57}$$

which is the usual form of the Ward–Takahashi identity.

**(d)** From Part (b),

$$\left[-\frac{1}{\alpha}\Box\left(\partial^\mu\frac{\delta W}{\delta J^\mu}\right)-\partial_\mu J^\mu - ie\left(-\eta\frac{\delta W}{\delta\eta}+\bar{\eta}\frac{\delta W}{\delta\bar{\eta}}\right)\right]=0.$$

Differentiating this with respect to $J_\nu(y)$ and setting $\bar{\eta}=\eta=J_\mu=0$, we get

$$-\frac{1}{\alpha}\Box\left(\partial^\mu\frac{\delta^2 W}{\delta J^\mu(x)\delta J^\nu(y)}\right)=\partial_x^\mu g_{\mu\nu}\delta^4(x-y). \tag{9.58}$$

Recall that the gauge field propagator is related to $W$ by

$$G_{\mu\nu}(x-y)=\langle 0|T(A_\mu(x)A_\nu(0))|0\rangle=(i)^2\frac{\delta^2 W}{\delta J^\mu(x)\delta J^\nu(y)}.$$

The Ward identity is then of the form

$$-\frac{1}{\alpha}\Box_x\partial_x^\mu\frac{\delta^2 W}{\delta J_\mu\delta J_\nu}=\partial_x^\mu g_{\mu\nu}\delta^4(x-y) \tag{9.59}$$

or in momentum space

$$\frac{i}{\alpha}k^2k^\mu\tilde{G}_{\mu\nu}(k)=k_\nu. \tag{9.60}$$

**Remark.** This relation is true to all order in $e$ and gives the result that the longitudinal part of $G_{\mu\nu}$ is not modified by the interaction. This can be seen as follows. Write $\tilde{G}_{\mu\nu}(k)$ as

$$\tilde{G}_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)G_T(k^2)+\frac{k_\mu k_\nu}{k^2}G_L(k^2). \tag{9.61}$$

Then the Ward identity implies that

$$\frac{i}{\alpha}k^2G_L(k^2)k_\nu=k_\nu \quad \text{or} \quad G_L(k^2)=\frac{-i\alpha}{k^2}.$$

This is just the lowest order result as seen in Problem 9.1 (with $\xi\to\alpha$).

## 9.7 Nilpotent BRST charges

The BRST (Becchi–Rouet–Stora–Tyutin) charge $Q$ is defined through the BRST transformations of a field

$$\delta\phi=\omega Q\phi \tag{9.62}$$

where $\omega$ is an arbitrary anticommuting Grassmann variable. From the BRS transformations given in CL-eqn (9.132), show that the BRST charge has the property of being nilpotent $Q^2\phi=0$ for (i) a gauge field $\phi=A_\mu^a$, (ii) a fermion field $\phi=\psi$, (iii) the ghost fields $\phi=\rho^a$ and $\sigma^a$.

**Hint.** For the case of ghost fields we need to use the equation of motion for the $\sigma^a$ to show $Q^2\rho^a = 0$.

**Solution to Problem 9.7**

From the BRST in CL-eqn (9.132), we can extract the properties of the BRST charges:

$$\delta A_\mu^a = \omega D_\mu \sigma^a \quad \Rightarrow \quad QA_\mu^a = D_\mu\sigma^a = \partial_\mu\sigma^a - g\varepsilon^{abc}\sigma^b A_\mu^a \tag{9.63}$$

$$\delta\psi = ig\omega(T^a\sigma^a)\psi \quad \Rightarrow \quad Q\psi = ig(T^a\sigma^a)\psi \tag{9.64}$$

$$\delta\rho^a = -\frac{i\omega}{\xi}(\partial^\mu A_\mu^a) \quad \Rightarrow \quad Q\rho^a = -\frac{i}{\xi}(\partial^\mu A_\mu^a) \tag{9.65}$$

$$\delta\sigma^a = -\frac{g}{2}\omega\varepsilon^{abc}\sigma^b\sigma^c \quad \Rightarrow \quad Q\sigma^a = -\frac{g}{2}\varepsilon^{abc}\sigma^b\sigma^c. \tag{9.66}$$

**(i) Gauge field**

$$\begin{aligned} Q^2 A_\mu^a &= Q(QA_\mu^a) = Q\left(\partial_\mu\sigma^a - g\varepsilon^{abc}\sigma^b A_\mu^c\right) \\ &= \partial_\mu(Q\sigma^a) - g\varepsilon^{abc}(Q\sigma^b)A_\mu^c + g\varepsilon^{abc}\sigma^b(QA_\mu^c). \end{aligned} \tag{9.67}$$

We now examine each term in turn:

$$\begin{aligned} \text{1st term} &= \partial_\mu(Q\sigma^a) = \partial_\mu\left(-\frac{g}{2}\varepsilon^{abc}\sigma^b\sigma^c\right) \\ &= -\frac{g}{2}\varepsilon^{abc}\left(\partial_\mu\sigma^b\sigma^c + \sigma^b\partial_\mu\sigma^c\right) = -g\varepsilon^{abc}\partial_\mu\sigma^b\sigma^c \\ \text{2nd term} &= g\varepsilon^{abc}\left(\frac{g}{2}\varepsilon^{bef}\sigma^e\sigma^f\right)A_\mu^c \\ \text{3rd term} &= g\varepsilon^{abc}\sigma^b(QA_\mu^c) = g\varepsilon^{abc}\sigma^b\left(\partial_\mu\sigma^c - g\varepsilon^{cef}\sigma^e A_\mu^f\right). \end{aligned}$$

The derivative terms cancel, we then have

$$Q^2 A_\mu^a = \frac{g^2}{2}\varepsilon^{abc}\varepsilon^{bef}\sigma^e\sigma^f A_\mu^c - g^2\varepsilon^{abc}\varepsilon^{cef}\sigma^b\sigma^e A_\mu^f. \tag{9.68}$$

Using the anticommuting property of the ghost field, the last term can be written, after relabelling indices, as

$$\begin{aligned} g^2\varepsilon^{aec}\varepsilon^{cbf}\sigma^e\sigma^b A_\mu^f &= \frac{g^2}{2}\left(\varepsilon^{abc}\varepsilon^{cef} - \varepsilon^{aec}\varepsilon^{cbf}\right)\sigma^b\sigma^e A_\mu^f \\ &= -\frac{g^2}{2}\varepsilon^{bec}\varepsilon^{acf}\sigma^b\sigma^e A_\mu^f = -\frac{g^2}{2}\varepsilon^{feb}\varepsilon^{abc}\sigma^f\sigma^e A_\mu^c, \end{aligned}$$

where to reach the last line, we have used the Jacobi identity

$$\varepsilon^{abc}\varepsilon^{cef} - \varepsilon^{aec}\varepsilon^{cbf} = -\varepsilon^{bec}\varepsilon^{acf}.$$

Finally we get

$$Q^2 A_\mu^a = \frac{g^2}{2}\varepsilon^{abc}\varepsilon^{bef}\sigma^e\sigma^f A_\mu^c - \frac{g^2}{2}\varepsilon^{feb}\varepsilon^{abc}\sigma^f\sigma^e A_\mu^c = 0. \tag{9.69}$$

**(ii) Fermion field**

$$\begin{aligned} Q^2\psi &= Q(igT^a\sigma^a\psi) = igT^a\left(Q\sigma^a\right)\psi - igT^a\sigma^a Q\psi \\ &= igT^a\left(-\frac{g}{2}\varepsilon^{abc}\sigma^b\sigma^c\right)\psi - igT^a\sigma^a(igT^b\sigma^b\psi). \end{aligned} \tag{9.70}$$

The second term can be shown to cancel the first term because it can be written as

$$g^2 T^a T^b\sigma^a\sigma^b\psi = \frac{g^2}{2}\left[T^a, T^b\right]\sigma^a\sigma^b\psi = \frac{ig^2}{2}\varepsilon^{abc}T^c\sigma^a\sigma^b\psi, \tag{9.71}$$

where we have used the fact that $\sigma$s anticommute with each other.

**(iii) Ghost fields**

$$\begin{aligned} Q^2\rho^a &= Q\left(-\frac{i}{\xi}\partial^\mu A_\mu^a\right) = -\frac{i}{\xi}\partial^\mu(QA_\mu^a) = -\frac{i}{\xi}\partial^\mu\left(\partial_\mu\sigma^a - g\varepsilon^{abc}\sigma^b A_\mu^c\right) \\ &= -\frac{i}{\xi}\left[\partial^2\sigma - g\varepsilon^{abc}\partial^\mu\left(\sigma^b A_\mu^c\right)\right]. \end{aligned} \tag{9.72}$$

The right-hand side vanishes because of the equation of motion for the $\sigma^a$ field as implied by CL-eqns (9.128) and (9.129) so that

$$Q^2\rho^a = 0. \tag{9.73}$$

To show that $Q^2\sigma^a = 0$:

$$\begin{aligned} Q^2\sigma^a &= Q\left(-\frac{g}{2}\varepsilon^{abc}\sigma^b\sigma^c\right) = -\frac{g}{2}\varepsilon^{abc}\left[(Q\sigma^b)\sigma^c - \sigma^b(Q\sigma^c)\right] \\ &= -\frac{g}{2}\varepsilon^{abc}\left[\left(-\frac{g}{2}\varepsilon^{bef}\sigma^e\sigma^f\right)\sigma^c - \sigma^b\left(-\frac{g}{2}\varepsilon^{cef}\sigma^e\sigma^f\right)\right] \\ &= \left(\frac{g}{2}\right)^2\varepsilon^{abc}\left[\varepsilon^{bef}\sigma^e\sigma^f\sigma^c - \sigma^c\left(\varepsilon^{bef}\sigma^e\sigma^f\right)\right] = 0. \end{aligned} \tag{9.74}$$

**Remark.** Since we have only used the antisymmetric property of the structure constant $\varepsilon^{abc}$ of the SU(2) group, the same calculation will go through if we replace $\varepsilon^{abc}$ by the more general structure constant $f^{abc}$ which is also totally antisymmetric.

## 9.8 BRST charges and physical states

Suppose an operator $Q$ is nilpotent, i.e. it has the property $Q^2 = 0$ and commutes with the Hamiltonian $[Q, H] = 0$.

**(a)** Show that we can divide the eigenstates of $H$ into three subspaces,

$$\mathcal{H}_1 = \{\psi_1;\, Q\psi_1 \neq 0\}, \qquad \mathcal{H}_2 = \{\psi_2;\, \psi_2 = Q\psi_1 \text{with } \psi_1 \in \mathcal{H}_1\},$$

$$\mathcal{H}_3 = \{\psi_3;\, Q\psi_3 = 0 \text{ but } \psi_3 \neq Q\psi_1\}. \tag{9.75}$$

**(b)** Show that the scalar product between any two states in $\mathcal{H}_2$ is zero:

$$\langle\psi_{2a}|\psi_{2b}\rangle = 0 \qquad \text{if} \quad \psi_{2a}, \psi_{2b} \in \mathcal{H}_2. \tag{9.76}$$

This implies that the states in $\mathcal{H}_2$ all have zero norm.

**(c)** Show that the scalar product between states in $\mathcal{H}_2$ and states in $\mathcal{H}_3$ is also zero.

**Remark.** If we select the physical states by imposing the condition

$$Q|\psi_{phys}\rangle = 0 \tag{9.77}$$

then the physical state is generally of the form

$$|\psi_{phys}\rangle = |\psi_3\rangle + |\psi_2\rangle \qquad \text{where} \quad |\psi_3\rangle \in \mathcal{H}_3, |\psi_2\rangle \in \mathcal{H}_2.$$

The results in (b) and (c) will imply that

$$\langle\psi'_{phys}|\psi_{phys}\rangle = \left(\langle\psi'_3| + \langle\psi'_2|\right)(|\psi_3\rangle + |\psi_2\rangle) = \langle\psi'_3|\psi_3\rangle. \tag{9.78}$$

This means that the zero norm states in $\mathcal{H}_2$ will not contribute to physical matrix elements and all important physics are contained in the space $\mathcal{H}_3$. The presence of $\mathcal{H}_1$ and $\mathcal{H}_2$ is to maintain the Lorentz and gauge invariance. This is exactly analogous to the Gupta–Bleuer quantization formalism of QED.

### Solution to Problem 9.8

**(a)** We can always separate the eigenstates of $H$ into two categories: $(i)$ $Q\psi \neq 0$ and $(ii)$ $Q\psi = 0$. The first category $(i)$ is just the space $\mathcal{H}_1$. In the category $(ii)$, we have two possibilities: it can be written in the form $\psi = Q\psi'$, so that $Q\psi = Q^2\psi' = 0$ (this corresponds to space $\mathcal{H}_2$), or it cannot be written as $Q\psi'$ but has the property that $Q\psi = 0$ (this corresponds to space $\mathcal{H}_3$).

**(b)**

$$\langle\psi_{2a}|\psi_{2b}\rangle = \langle\psi_{1a}|Q|\psi_{2b}\rangle = 0 \tag{9.79}$$

because

$$Q|\psi_{2b}\rangle = 0 \qquad \text{and} \qquad |\psi_{2a}\rangle = Q|\psi_{1a}\rangle. \tag{9.80}$$

**(c)**

$$\langle\psi_{2a}|\psi_{3b}\rangle = \langle\psi_{1a}|Q|\psi_{3b}\rangle = 0. \tag{9.81}$$

# 10 Quantum chromodynamics

## 10.1 Colour factors in QCD loops

In QCD loop calculations we often encounter the some $SU_c(3)$ group theoretical factors. In this problem you will be asked to calculate such factors for the general case $SU(n)$ rather than the $n = 3$ special situation of three colours.

In the quark loop diagram of Fig. 10.1(a), we have the trace factor for the quadratic product of

$$Tr\left(T_F^a T_F^b\right) = t_2(F)\delta_{ab} \tag{10.1}$$

where $a = 1, 2, \ldots, (n^2 - 1)$ and $T_R^a$ stands for the $SU(n)$ generator in the representation $R$. For the present case of the quark loop, $R$ is the fundamental representation $F$,

$$T_F^a = \frac{\lambda^a}{2}, \tag{10.2}$$

with $\{\lambda^a\}$ being the usual $n \times n$ hermitian traceless matrices, and the above trace becomes

$$\sum_{i,j} \left(\frac{\lambda^a}{2}\right)_{ij} \left(\frac{\lambda^b}{2}\right)_{ji} = \frac{1}{2}\delta_{ab}. \tag{10.3}$$

where $i = 1, 2, \ldots, n$. Thus the trace factor $t_2(F) = \frac{1}{2}$ normalizes the (bare) QCD coupling $g$.

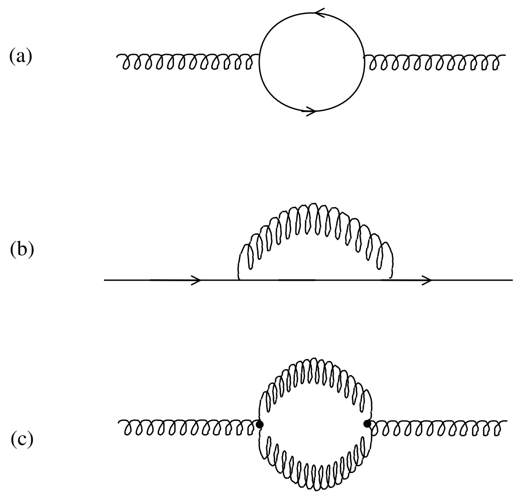



FIG. 10.1. Quark and gluon loops in QCD.

In the quark self-energy diagram of Fig. 10.1(b), we encounter the group theoretical factor of

$$\sum_a \left(T_F^a T_F^a\right)_{ij} = C_2(F)\delta_{ij} \tag{10.4}$$

where $C_2(R)$ is the eigenvalue of the quadratic Casimir operator in the representation $R$. For the present case of quarks in the fundamental representation, we have

$$\tfrac{1}{4}\sum_a (\lambda^a\lambda^a)_{ij} = C_2(F)\delta_{ij}. \tag{10.5}$$

[Here we are following the more commonly used notation of $C_2(F)$, rather than $s_2(V)$ as in CL-text.]

In the gluon loop of Fig. 10.1(c), the sum over colour indices can be represented either as a trace, like eqn (10.1),

$$Tr\left(T_A^b T_A^c\right) = t_2(A)\delta_{bc}, \tag{10.6}$$

or as a Casimir operator, like eqn (10.4),

$$\sum_a \left(T_A^a T_A^a\right)_{bc} = C_2(A)\delta_{bc}, \tag{10.7}$$

where $T_A^a$ is the generator in the adjoint representation $A$, as is appropriate for the gluon gauge field,

$$\left(T_A^a\right)_{bc} = C_{abc} \tag{10.8}$$

where $C_{abc}$ is the structure constant of the SU($n$) algebra. Since $C_{abc}$ is totally antisymmetric, it is clear that the above two expressions are equivalent:

$$t_2(A) = C_2(A). \tag{10.9}$$

**(a)** Show that for SU($n$) the value for $C_2(A)$ is

$$C_2(F) = \frac{1}{2n}(n^2-1). \tag{10.10}$$

**(b)** Show that

$$C_2(A) = n \tag{10.11}$$

which is denoted by $t_2(V)$ in CL-text.

### Solution to Problem 10.1

**(a)** From the identity CL-eqn (4.134),

$$\sum_d (\lambda_d)_{ij}(\lambda_d)_{kl} = 2\left(\delta_{il}\delta_{jk} - \frac{1}{n}\delta_{ij}\delta_{kl}\right), \tag{10.12}$$

we get

$$\sum_a (\lambda_a\lambda_a)_{il} = \sum_a (\lambda_a)_{ij}(\lambda_a)_{jl} = 2\left(\delta_{il}\delta_{jk} - \frac{1}{n}\delta_{ij}\delta_{kl}\right)\delta_{jk}$$
$$= 2\left(n\delta_{il} - \frac{1}{n}\delta_{il}\right) = \frac{2(n^2-1)}{n}\delta_{il}. \tag{10.13}$$

Thus

$$\sum_a \left[T_F^a T_F^a\right]_{ij} = \tfrac{1}{4}\sum_a (\lambda_a\lambda_a)_{ij} = \frac{(n^2-1)}{2n}\delta_{ij}. \tag{10.14}$$

Namely,

$$C_2(F) = \frac{1}{2n}(n^2-1). \tag{10.15}$$

**(b)** From the SU($n$) Lie algebra

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = iC^{abc}\frac{\lambda_c}{2} \tag{10.16}$$

and the normalization $Tr(\lambda_a\lambda_b) = 2\delta_{ab}$, we can write

$$C^{abc} = \frac{-i}{4}Tr(\lambda_c[\lambda_a, \lambda_b]). \tag{10.17}$$

Then we have

$$C^{acd}C^{bcd} = -\frac{1}{16}Tr(\lambda_d[\lambda_b, \lambda_c])Tr(\lambda_d[\lambda_a, \lambda_c])$$
$$= -\frac{1}{16}(\lambda_d)_{ij}[\lambda_b, \lambda_c]_{ji}(\lambda_d)_{kl}[\lambda_a, \lambda_c]_{lk}. \tag{10.18}$$

Using the identity (10.12), we get

$$C^{acd}C^{bcd} = -\frac{1}{8}Tr([\lambda_b, \lambda_c][\lambda_a, \lambda_c]) - \frac{1}{n}Tr([\lambda_b, \lambda_c])Tr([\lambda_a, \lambda_c]). \tag{10.19}$$

The second term vanishes because $Tr([A, B]) = 0$ for any two matrices $A$ and $B$. Thus

$$C^{acd}C^{bcd} = -\frac{1}{8}Tr\left[(\lambda_b\lambda_c - \lambda_c\lambda_b)(\lambda_a\lambda_c - \lambda_c\lambda_a)\right]$$
$$= -\frac{1}{8}Tr\left(\lambda_b\lambda_c\lambda_a\lambda_c - \lambda_b\lambda_a\lambda_c\lambda_c - \lambda_b\lambda_c\lambda_c\lambda_a\right.$$
$$\left.+ \lambda_c\lambda_b\lambda_c\lambda_a\right). \tag{10.20}$$

The first term can be calculated as

$$
\begin{aligned}
\text{1st term} &= Tr(\lambda_b\lambda_c\lambda_a\lambda_c) = (\lambda_b)_{ij}(\lambda_c)_{jk}(\lambda_a)_{kl}(\lambda_c)_{li} \\
&= (\lambda_b)_{ij}(\lambda_a)_{kl}2\left(\delta_{ij}\delta_{kl} - \frac{1}{n}\delta_{jk}\delta_{il}\right) \\
&= 2Tr(\lambda_b)Tr(\lambda_a) - \frac{2}{n}Tr(\lambda_b\lambda_a) = -\frac{4}{n}\delta_{ab},
\end{aligned}
\tag{10.21}
$$

and the second term is

$$
\begin{aligned}
\text{2nd term} &= Tr(\lambda_b\lambda_a\lambda_c\lambda_c) = (\lambda_b)_{ij}(\lambda_a)_{jk}(\lambda_c)_{kl}(\lambda_c)_{li} \\
&= (\lambda_b)_{ij}(\lambda_a)_{jk}2\left(\delta_{ll}\delta_{ki} - \frac{1}{n}\delta_{kl}\delta_{il}\right) \\
&= 2nTr(\lambda_b\lambda_a) - \frac{2}{n}Tr(\lambda_b\lambda_a) = 4\left(n - \frac{1}{n}\right)\delta_{ab}.
\end{aligned}
\tag{10.22}
$$

Similarly,

$$\text{3rd term} = 4\left(n - \frac{1}{n}\right)\delta_{ab}, \qquad \text{4th term} = -\frac{4}{n}\delta_{ab}. \tag{10.23}$$

We then have

$$C^{acd}C^{bcd} = -\frac{1}{8}2\left[-\frac{4}{n} - 4\left(n - \frac{1}{n}\right)\right]\delta_{ab} = n\delta_{ab} \tag{10.24}$$

Namely,

$$C_2(A) = t_2(A) = n. \tag{10.25}$$

## 10.2 Running gauge coupling in two-loop

In QCD, the $\beta$-function in two-loop is of the form,

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \cdots \tag{10.26}$$

where

$$\beta_0 = \frac{1}{(4\pi)^2}\left(11 - \frac{2}{3}N_f\right) \quad \text{and} \quad \beta_1 = \frac{1}{(4\pi)^4}\left(102 - \frac{38}{3}N_f\right). \tag{10.27}$$

Show that the effective coupling constant $\bar{g}$ defined by (with $t = \frac{1}{2}\ln Q^2/\mu^2$)

$$\frac{d\bar{g}(g,t)}{dt} = \beta(\bar{g}) \qquad \text{with} \quad \bar{g}(g,0) = g \tag{10.28}$$

can be written as

$$\bar{g}^2 = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)} + \cdots\right]. \tag{10.29}$$

**Solution to Problem 10.2**

The effective coupling is defined by the equation

$$\frac{d\bar{g}}{dt} = \beta(\bar{g}) = -\beta_0 \bar{g}^3 - \beta_1 \bar{g}^5. \tag{10.30}$$

Let us introduce $\lambda = \bar{g}^2$, then

$$\frac{d\lambda}{dt} = -2\left(\beta_0 \lambda^2 + \beta_1 \lambda^3\right) \tag{10.31}$$

or

$$\int dt = -\frac{1}{2} \int_{g^2}^{\bar{g}^2} \frac{d\lambda}{\lambda^2} \frac{1}{(\beta_0 + \beta_1 \lambda)} \tag{10.32}$$

or

$$t = -\frac{1}{2} \left[ -\frac{1}{\beta_0 \lambda} + \frac{\beta_1}{\beta_0^2} \ln \frac{\beta_0 + \beta_1 \lambda}{\lambda} \right]_{g^2}^{\bar{g}^2}. \tag{10.33}$$

Write $t = \frac{1}{2} \ln Q^2/\mu^2$, then we get

$$\beta_0 \ln Q^2 - \beta_0 \ln \mu^2 = \frac{1}{\bar{g}^2} - \frac{1}{g^2} - \frac{\beta_1}{\beta_0} \left[ \ln \frac{\left(\beta_0 + \beta_1 \bar{g}^2\right)}{\bar{g}^2} - \ln \frac{\left(\beta_0 + \beta_1 g^2\right)}{g^2} \right].$$

Combining all the $Q^2$ independent terms, we define the scale parameter $\Lambda$ by

$$\beta_0 \ln \Lambda^2 \equiv \beta_0 \ln \mu^2 - \frac{1}{g^2} + \frac{\beta_1}{\beta_0} \ln \frac{\left(\beta_0 + \beta_1 g^2\right)}{g^2} \tag{10.34}$$

so that we have a simpler relation,

$$\beta_0 \ln \frac{Q^2}{\Lambda^2} \equiv \frac{1}{\bar{g}^2} - \frac{\beta_1}{\beta_0} \ln \frac{\left(\beta_0 + \beta_1 \bar{g}^2\right)}{\bar{g}^2}. \tag{10.35}$$

We can solve this for $\bar{g}^2$ by iteration. To lowest order in $g$,

$$\left( \frac{1}{\bar{g}^2} \right)_0 = \beta_0 \ln \frac{Q^2}{\Lambda^2}. \tag{10.36}$$

The second factor in eqn (10.35) can be approximated as

$$\begin{aligned} \ln \frac{(\beta_0 + \beta_1 \bar{g}^2)}{\bar{g}^2} &= \ln \left( \beta_1 + \frac{\beta_0}{\bar{g}^2} \right) \simeq \ln \left( \beta_1 + \beta_0^2 \ln \frac{Q^2}{\Lambda^2} \right) \\ &\simeq \ln \ln \frac{Q^2}{\Lambda^2} \qquad \text{for} \quad \text{large } \frac{Q^2}{\Lambda^2}. \end{aligned} \tag{10.37}$$

To next order in $g$, we have

$$\frac{1}{\bar{g}^2} = \beta_0 \ln \frac{Q^2}{\Lambda^2} + \frac{\beta_1}{\beta_0} \ln \ln \frac{Q^2}{\Lambda^2} = \left( \beta_0 \ln \frac{Q^2}{\Lambda^2} \right) \left[ 1 + \frac{\beta_1}{\beta_0^2} \frac{\ln \ln (Q^2/\Lambda^2)}{\ln (Q^2/\Lambda^2)} \right]$$

or

$$\bar{g}^2 = \frac{1}{(\beta_0 \ln Q^2/\Lambda^2)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right]. \tag{10.38}$$

## 10.3 Cross-section for three-jet events

Consider the process (see Peskin and Schroeder 1995 for further discussion)

$$e^+(p') + e^-(p) \to q(k_1) + \bar{q}(k_2) + g(k_3). \tag{10.39}$$

**(a)** Show that the three-body phase space can be written as

$$\begin{aligned}\rho &= \int \frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3}(2\pi)^4\delta^4(q - k_1 - k_2 - k_3)\\ &= \frac{q^2}{128\pi^3}\int dx_1\, dx_2\end{aligned} \tag{10.40}$$

where $x_i = 2k_i \cdot q/q^2$, with $i = 1, 2, 3$ and $q_\mu = p_\mu + p'_\mu$. Find the region of integration for $x_1$ and $x_2$, for the case where quarks are massless but gluon has a mass $\mu$.

**(b)** Show that the amplitude for this process can be written as

$$\mathcal{M} = e^2 g[\bar{v}(p')\gamma^\mu u(p)]\frac{1}{(q^2 + i\varepsilon)}[\bar{u}(k_1)\Lambda_{\lambda\mu}v(k_2)]\varepsilon^\lambda(k_3)Q_q \tag{10.41}$$

with the fractional charge of the quark $Q_u = 2/3$ and $Q_d = -1/3$, etc., and

$$\Lambda_{\lambda\mu} = \gamma_\lambda \frac{-1}{\not{k}_1 + \not{k}_2}\gamma_\mu + \gamma_\mu \frac{1}{\not{k}_3 + \not{k}_2}\gamma_\lambda. \tag{10.42}$$

**(c)** Show that, in the limit of massless quarks and gluon, the differential cross-section can be written as

$$\frac{d^2\sigma}{dx_1 dx_2} = \frac{4\pi\alpha^2}{3s} 8Q_q^2\left(\frac{\alpha_s}{2\pi}\right)\left[\frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}\right] \tag{10.43}$$

where $s = (p + p')^2$.

**(d)** Show that in the integration over $x_1$ and $x_2$ there are infrared divergences as $\mu \to 0$, corresponding to configurations where the gluon is collinear with the quarks, $q$ or $\bar{q}$—the *collinear divergence*.

### Solution to Problem 10.3

**(a)** Integrating over the three-dimensional $\delta$-function, we get

$$\rho = \int \frac{d^3k_1\, d^3k_2}{(2\pi)^5 2\omega_1 2\omega_2 2\omega_3}\delta(q_0 - \omega_1 - \omega_2 - \omega_3). \tag{10.44}$$

Choose a frame such that $\mathbf{q} = 0$, which gives $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$. The gluon energy can be written as

$$\omega_3 = \left[\mu^2 + (\mathbf{k}_3)^2\right]^{1/2} = \left[\mu^2 + (\mathbf{k}_1 + \mathbf{k}_2)^2\right]^{1/2} \tag{10.45}$$

or

$$\omega_3^2 = \mu^2 + k_1^2 + k_2^2 + 2k_1k_2\cos\theta \quad \Rightarrow \quad \omega_3 d\omega_3 = k_1k_2 d(\cos\theta) \tag{10.46}$$

with $\theta$ the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. Also,

$$d^3k_1\, d^3k_2 = (4\pi)(2\pi)\, d(\cos\theta) k_1^2\, dk_1 k_2^2\, dk_2 = 8\pi^2 \omega_3\, d\omega_3 \omega_1\, d\omega_1 \omega_2\, d\omega_2 \tag{10.47}$$

where we have used $\omega_i\, d\omega_i = k_i\, dk_i$. It then follows that

$$\begin{aligned}\rho &= \frac{8\pi^2}{(2\pi)^5} \int \frac{\omega_3\, d\omega_3 \omega_1\, d\omega_1 \omega_2\, d\omega_2}{2\omega_1 2\omega_2 2\omega_3} \delta(q_0 - \omega_1 - \omega_2 - \omega_3) \\ &= \frac{1}{32\pi^3} \int d\omega_3\, \delta(q - \omega_1 - \omega_2 - \omega_3) d\omega_1\, d\omega_2 = \frac{1}{32\pi^3} \int d\omega_1\, d\omega_2\end{aligned}$$

For $\mathbf{q} = 0$, we have

$$x_1 = \frac{2k_1 \cdot q}{q^2} = \frac{2\omega_1}{q_0} \quad \Rightarrow \quad dx_1 dx_2 = \frac{4}{q_0^2} d\omega_1\, d\omega_2 \tag{10.48}$$

and

$$\rho = \frac{q^2}{128\pi^3} \int dx_1\, dx_2. \tag{10.49}$$

From $\omega_1 = (m^2 + \mathbf{k}_1^2)^{1/2}$, the minimum for $x_1$ is $2m/q_0$ which goes to 0 for $m = 0$. Similarly, $\omega_3$ has a minimum at $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 = 0$, which implies that $\omega_1 = \omega_2$, and recall, for a massive gluon, the minimum is at $\omega_3 = \mu$. It is easy to see that this configuration gives a maximum value for $\omega_1$, or $\omega_2$. From energy conservation, the maximum for $\omega_1$ is

$$q_0 = \omega_1 + \omega_2 + \omega_3 = \mu + 2\omega_1 \quad \text{or} \quad 2\omega_1 = q_0 - \mu$$

or

$$x_1 = 1 - \frac{\mu}{q_0} = 1 - \frac{\mu}{\sqrt{q^2}}. \tag{10.50}$$

Thus the range of integration is

$$0 \le x_1 \le 1 - \frac{\mu}{\sqrt{q^2}}, \qquad 0 \le x_2 \le 1 - \frac{\mu}{\sqrt{q^2}}. \tag{10.51}$$

**(b)**

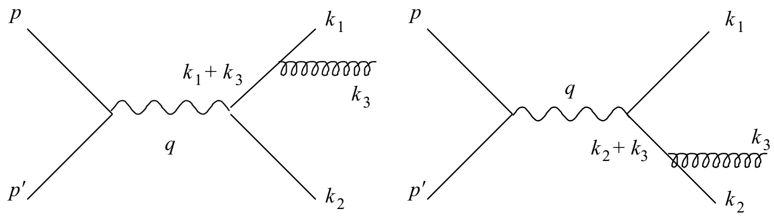



FIG. 10.2. Gluon bremsstrahlung in $e^+e^-$ annihilation.

From the Feynman rule, the amplitude is given by

$$\begin{aligned}\mathcal{M} &= -i[\bar{v}(p')(-ie\gamma_\mu)u(p)]\frac{1}{(q^2+i\varepsilon)}\Bigg[\bar{u}(k_1)(-ig\gamma_\lambda)\\ &\quad\times\frac{i}{\not{k}_1+\not{k}_3}(-ie\gamma^\mu)v(k_2)+\bar{u}(k_1)(-ie\gamma^\mu)\\ &\quad\times\frac{-i}{\not{k}_3+\not{k}_2}(-ig\gamma_\lambda)v(k_2)\Bigg]Q_q\varepsilon^\lambda(k_3)\\ &= \frac{e^2g}{q^2}[\bar{v}(p')\gamma^\mu u(p)][\bar{u}(k_1)\Lambda_{\lambda\mu}v(k_2)]\varepsilon^\lambda(k_3)Q_q\end{aligned}$$

with

$$\Lambda_{\lambda\mu} = \gamma_\lambda\frac{-1}{\not{k}_1+\not{k}_3}\gamma_\mu+\gamma_\mu\frac{1}{\not{k}_3+\not{k}_2}\gamma_\lambda. \tag{10.52}$$

**(c)** We can write

$$\Lambda_{\lambda\mu} = -\gamma_\lambda\frac{\not{k}_1+\not{k}_3}{(k_1+k_3)^2}\gamma_\mu+\gamma_\mu\frac{\not{k}_2+\not{k}_3}{(k_2+k_3)^2}\gamma_\lambda. \tag{10.53}$$

The denominators can be simplified:

$$(k_1+k_3)^2 = (q-k_2)^2 = q^2-2k_2\cdot q = q^2(1-x_2) \tag{10.54}$$

and

$$(k_2+k_3)^2 = q^2(1-x_1). \tag{10.55}$$

Then

$$\Lambda_{\lambda\mu} = \frac{-1}{q^2(1-x_2)}\gamma_\lambda(\not{k}_1+\not{k}_3)\gamma_\mu+\frac{1}{q^2(1-x_1)}\gamma_\mu(\not{k}_2+\not{k}_3)\gamma_\lambda. \tag{10.56}$$

The differential cross-section is then

$$\begin{aligned}d\sigma &= \frac{1}{4(p\cdot p')}\left(\frac{1}{4}\sum_{spin}|\mathcal{M}|^2\right)\frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3}\\ &\quad\times(2\pi)^4\delta^4(q-k_1-k_2-k_3)\\ &= \frac{1}{2(q)^2}\left(\frac{1}{4}\sum_{spin}|\mathcal{M}|^2\right)\rho \qquad \text{with} \quad \rho = \frac{q^2}{128\pi^3}\int dx_1\,dx_2.\end{aligned} \tag{10.57}$$

The calculation of the matrix element is straightforward but tedious. After using the relation

$$\sum_s \varepsilon_\lambda(s,k_3)\varepsilon_\beta(s,k_3) = -g_{\lambda\beta} + \frac{k_{3\lambda}k_{3\beta}}{k^2} \tag{10.58}$$

(the $k_{3\lambda}k_{3\beta}$ term in the photon polarization sum will not contribute because of gauge invariance) we have

$$\begin{aligned}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{1}{4}\left(\frac{e^4g^2}{q^4}\right) Tr\left(\not{p}'\gamma_\mu \not{p}\,\gamma_\nu\right) Tr\left(\not{k}_1\Lambda_{\lambda\mu}\not{k}_2\Lambda_{\nu\lambda}\right)\\ &= \frac{e^4g^2}{4q^4} l_{\mu\nu}G^{\mu\nu}\end{aligned} \tag{10.59}$$

where

$$l_{\mu\nu} = Tr\left(\not{p}'\gamma_\mu\not{p}\,\gamma_\nu\right) = 4\left(p'_\mu p_\nu + p_\mu p'_\nu - g_{\mu\nu}\,p\cdot p'\right) \tag{10.60}$$

$$G^{\mu\nu} = Tr\left(\not{k}_1\Lambda_{\lambda\mu}\not{k}_2\Lambda_{\nu\lambda}\right). \tag{10.61}$$

Writing the three-body phase space as

$$\rho = \frac{q^2}{128\pi^3}\int dx_1\,dx_2 = \int d\rho_3, \tag{10.62}$$

we have

$$\int d\rho_3 \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \kappa l_{\mu\nu}\int G^{\mu\nu}\,d\rho_3 \qquad \text{with} \quad \kappa = \frac{e^4g^2}{4q^4}. \tag{10.63}$$

The gauge invariance implies that

$$q_\mu G^{\mu\nu} = 0, \qquad q_\nu G^{\mu\nu} = 0, \qquad q_\mu l^{\mu\nu} = 0, \qquad q_\nu l^{\mu\nu} = 0. \tag{10.64}$$

The tensor $G^{\mu\nu}$, after the integration, can depend only on $q_\mu$. Thus,

$$\int G^{\mu\nu} d\rho_3 = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) G(q^2) \tag{10.65}$$

or

$$G(q^2) = \frac{1}{3}\int G^{\mu\nu}g_{\mu\nu}d\rho_3 \tag{10.66}$$

and

$$l_{\mu\nu}\int G^{\mu\nu}\,d\rho_3 = l_{\mu\nu}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)G(q^2) = l_{\mu\nu}g^{\mu\nu}\frac{1}{3}\int G^{\alpha\beta}g_{\alpha\beta}\,d\rho_3.$$

From eqn (10.60) we have

$$l_{\mu\nu}g^{\mu\nu} = -8p'\cdot p = -4q^2. \tag{10.67}$$

The quark tensor is somewhat complicated:

$$
\begin{aligned}
\bar{G} &= G^{\alpha\beta} g_{\alpha\beta} = Tr\left(\not{k}_1 \Lambda_{\lambda\mu} \not{k}_2 \Lambda^{\mu\lambda}\right) \\
&= Tr\left\{\not{k}_1 \left[\frac{-1}{q^2(1-x_2)}\gamma_\lambda(\not{k}_1+\not{k}_3)\gamma_\mu + \frac{1}{q^2(1-x_1)}\gamma_\mu(\not{k}_2+\not{k}_3)\gamma_\lambda\right]\right. \\
&\quad \left. \times \not{k}_2 \left[\frac{-1}{q^2(1-x_2)}\gamma^\mu(\not{k}_1+\not{k}_3)\gamma^\lambda + \frac{1}{q^2(1-x_1)}\gamma^\lambda(\not{k}_2+\not{k}_3)\gamma^\mu\right]\right\}.
\end{aligned} \tag{10.68}
$$

The trace in the term containing $(1-x_2)^2$ in the denominator is

$$
\begin{aligned}
&Tr\{\not{k}_1\gamma_\lambda(\not{k}_1+\not{k}_3)\gamma_\mu\not{k}_2\gamma^\mu(\not{k}_1+\not{k}_3)\gamma^\lambda\} \\
&\quad = -2Tr\{\not{k}_1(\not{k}_1+\not{k}_3)\gamma_\mu\not{k}_2\gamma^\mu(\not{k}_1+\not{k}_3)\} \\
&\quad = 4Tr(\not{k}_1\not{k}_3\not{k}_2\not{k}_3) = 16(2(k_1\cdot k_3)(k_2\cdot k_3)) \\
&\quad = 8q^4(1-x_1)(1-x_2)
\end{aligned} \tag{10.69}
$$

where we have set $k_3^2 = 0$ and used the relations

$$
k_1\cdot k_3 = \frac{1}{2}(k_1+k_3)^2 = \frac{1}{2}(q-k_2)^2 = \frac{q^2}{2}(1-x_2), \qquad k_2\cdot k_3 = \frac{q^2}{2}(1-x_1).
$$

It is clear that the trace in the term containing $(1-x_1)^2$ in the denominator is exactly the same as above. The trace in the term containing $(1-x_2)(1-x_1)$ in the denominator is

$$
\begin{aligned}
&-Tr\{\not{k}_1\gamma_\lambda(\not{k}_1+\not{k}_3)\gamma_\mu\not{k}_2\gamma^\lambda(\not{k}_2+\not{k}_3)\gamma^\mu\} \\
&\quad = 2Tr\{\not{k}_1\not{k}_2\gamma_\mu(\not{k}_1+\not{k}_3)(\not{k}_2+\not{k}_3)\gamma^\mu\} \\
&\quad = 8(k_1+k_3)\cdot(k_2+k_3)4k_1\cdot k_2 \\
&\quad = 32(k_1\cdot k_2 + k_1\cdot k_3 + k_3\cdot k_2)\frac{q^2}{2}(1-x_3) \\
&\quad = 8q^4[(1-x_3)+(1-x_2)+(1-x_1)](1-x_3) \\
&\quad = 8q^4(1-x_3) = -8q^4(1-x_1-x_2)
\end{aligned} \tag{10.70}
$$

where we have used

$$
x_1+x_2+x_3 = \frac{2(k_1+k_2+k_3)\cdot q}{q^2} = \frac{2q^2}{q^2} = 2. \tag{10.71}
$$

Putting all these together, we get

$$
\begin{aligned}
\bar{G} &= 8\left\{\frac{(1-x_1)}{(1-x_2)} + \frac{(1-x_2)}{(1-x_1)} - \frac{2(1-x_1-x_2)}{(1-x_1)(1-x_2)}\right\} \\
&= \frac{8}{(1-x_1)(1-x_2)}\{(1-x_1)^2+(1-x_2)^2-2(1-x_1-x_2)\} \\
&= \frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}.
\end{aligned} \tag{10.72}
$$

The differential cross-section is then

$$\begin{aligned} d\sigma &= \frac{1}{2q^2}\frac{e^4 g^2}{4q^4}(4q^2)\frac{1}{3}\int \frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}dx_1\,dx_2\frac{q^2}{128\pi^3}Q_q^2 \\ &= \frac{2\alpha^2\alpha_s^2}{3q^2}Q_q^2\int\frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}dx_1\,dx_2. \end{aligned} \tag{10.73}$$

**Remark.** There should also be a colour factor of

$$\sum_a Tr\left(\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right) = \frac{2}{4}\sum_a \delta_{aa} = 4. \tag{10.74}$$

We then have

$$\frac{d^2\sigma}{dx_1\,dx_2} = \frac{8\alpha^2\alpha_s^2}{3q^2}Q_q^2\frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}. \tag{10.75}$$

**(d)** The range of integration is

$$0 \le x_1, x_2 \le 1 - \frac{\mu}{\sqrt{q^2}}. \tag{10.76}$$

Thus as $\mu \to 0$, the upper limit approaches 1 and the integrations over $x_1$ and $x_2$ are infrared divergent. [For the case $\mu \neq 0$, this gives terms of the form $(\ln \mu^2/q^2)^2$]. The region $x_1 \to 1$ corresponds to a configuration where the quark $q$ has maximum energy while $\bar{q}$ and gluon both are moving in the same direction, i.e. $\bar{q}$ and gluon are collinear.

## 10.4 Operator-product expansion of two currents

Consider the operator of the form

$$t_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x}T(J_\mu(x)J_\nu(0)) \tag{10.77}$$

where $J_\mu(x)$ is the electromagnetic current. The operator-product expansion can be written in the symbolic form as

$$t_{\mu\nu}(q) \sim \sum_i C_i^{\mu\nu}(q)O^i(0) \tag{10.78}$$

where $C_i^{\mu\nu}(q)$s are the Fourier transform of the Wilson coefficient in the coordinate space and the local operator has the general form of

$$O_d^{\mu_1\mu_2\cdots\mu_n}(0) \tag{10.79}$$

which is completely symmetric and traceless in $(\mu_1\mu_2\cdots\mu_n)$. The dimension of the operator is $d$, and spin is $n$.

**(a)** From dimensional analysis, show that we can write the forward matrix elements as

$$\langle p, s|O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)|p, s\rangle = M^{d-n-2}S[p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}]\alpha_{nV} \tag{10.80}$$

$$\langle p, s|O_{d,A}^{\mu_1\mu_2\cdots\mu_n}(0)|p, s\rangle = M^{d-n-2}S[s^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}]\alpha_{nA} \tag{10.81}$$

where subscript $V$ means that the operator $O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)$ has the same parity as the product of $n$ polar vectors, e.g. $x^{\mu_1}x^{\mu_2}\cdots x^{\mu_n}$, while $A$ denotes the axial type of operator $O_{d,A}^{\mu_1\mu_2\cdots\mu_n}(0)$ which has the same parity as the product of one axial vector and $(n-1)$ polar vectors. The state $|p, s\rangle$ has momentum $p^\mu$ ($p^2 = M^2$) and the polarization is described by a polarization vector $s^\mu$ and is normalized as

$$\langle p', s'|p, s\rangle = 2E_p\delta_{ss'}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'). \tag{10.82}$$

The operation $S[\cdots]$ projects out the completely symmetric traceless components. Also $\alpha_{nV}$, $\alpha_{nA}$ are dimensionless constants.

**(b)** Show that the corresponding Wilson coefficients, which give the leading contribution in the scaling limit, have the structure

$$C_1^{\mu\nu\mu_1\cdots\mu_n}(q) = -g^{\mu\nu}S[q^{\mu_1}q^{\mu_2}\cdots q^{\mu_n}](-q^2)^{(2-d-n)/2}C_{1n}(g) \tag{10.83}$$

for the $F_1$ structure function (see CL-Chapter 7 for the definition),

$$C_2^{\mu\nu\mu_1\cdots\mu_n}(q) = g^{\mu\mu_1}g^{\nu\mu_2}S[q^{\mu_3}q^{\mu_4}\cdots q^{\mu_n}](-q^2)^{(4-d-n)/2}C_{2n}(g) \tag{10.84}$$

for the $F_2$ structure function, and

$$C_3^{\mu\nu\mu_1\cdots\mu_n}(q) = \varepsilon^{\mu\nu\mu_1\alpha}S\left[q_\alpha q^{\mu_2}\cdots q^{\mu_n}\right](-q^2)^{(2-d-n)/2}C_{3n}(g) \tag{10.85}$$

for the $g_1$ spin-dependent structure function. $C_{i\,n}(g)$s are dimensionless numbers depending only on the coupling constant $g$.

**(c)** Show that

$$C_1^{\mu\nu\mu_1\cdots\mu_n}(q)\langle p, s|O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)|p, s\rangle = -g^{\mu\nu}\left(\frac{1}{x}\right)^n\left(\frac{-q^2}{M^2}\right)^{(2-d+n)/2} \tag{10.86}$$

$$C_2^{\mu\nu\mu_1\cdots\mu_n}(q)\langle p, s|O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)|p, s\rangle = \frac{p^\mu p^\nu}{M^2}\left(\frac{1}{x}\right)^{n-2}\left(\frac{-q^2}{M^2}\right)^{(-d+n)/2} \tag{10.87}$$

### Solution to Problem 10.4

**(a)** From the normalization of state

$$\langle p, s|p', s'\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')2E_p \tag{10.88}$$

we see that the physical state $|p, s\rangle$ has dimension $-1$. Thus the matrix element $\langle p, s|O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)|p, s\rangle$ has dimension $(d-2)$. Since $p_\mu$ is the only polar vector this matrix element can depend on, the Lorentz indices $\mu_1 \cdots \mu_n$ in $O_{d,V}$ are taken up by $p^{\mu_1} p^{\mu_1} \cdots p^{\mu_n}$. The term of the form $g^{\mu_1\mu_2} p^{\mu_3} p^{\mu_4} \cdots p^{\mu_n}$ is not traceless and will give a non-leading term in the scaling limit. From these considerations, we see that the general structure for the matrix element is

$$\langle p, s | O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0) | p, s\rangle = M^{d-n-2} S[p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}] \alpha_{nV} \tag{10.89}$$

where $\alpha_{nV}$ is a dimensionless constant. The matrix elements of the axial operators can be obtained similarly.

**(b)** The Wilson coefficient of the operator $O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)$ must have the Lorentz structure $C_{\mu\nu\mu_1\mu_2\cdots\mu_n}$. Since $J_\mu(x)$ has dimension three, we see that $t_{\mu\nu}(q)$ has dimension two. Since $O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)$ has dimension $d$, the Wilson coefficient $C_{\mu\nu\mu_1\mu_2\cdots\mu_n}(q)$ will have dimension $(2-d)$. The structure functions $W_1$, $W_2$, and $G_1$ are defined by

$$\frac{1}{2M}\langle p, s | t_{\mu\nu} | p, s\rangle = -g_{\mu\nu} T_1 + \frac{p_\mu p_\nu}{M^2} T_2 + \varepsilon_{\mu\nu\alpha\beta} s^\alpha q^\beta G_1 + \cdots \tag{10.90}$$

and

$$W_i = \frac{1}{\pi}\,\mathrm{Im}\, T_i \qquad \text{and} \qquad g_1 = \mathrm{Im}\, G_1. \tag{10.91}$$

Thus for structure function $W_1$, the Wilson coefficient is of the form

$$C_1^{\mu\nu\mu_1\cdots\mu_n}(q) = -g^{\mu\nu} S[q^{\mu_1} q^{\mu_2} \cdots q^{\mu_n}](-q^2)^{(2-d-n)/2} C_{1n}(g) \tag{10.92}$$

where $C_{1n}$ is a dimensionless constant and can depend only on the coupling constant $g$.

Similarly, for the structure function $W_2$, the Wilson coefficient is of the form

$$\begin{aligned} C_2^{\mu\nu\mu_1\cdots\mu_n}(q) = \big\{ & g^{\mu\mu_1} g^{\mu\mu_2} S[q^{\mu_3} q^{\mu_4} \cdots q^{\mu_n}](-q^2)^{(4-d-n)/2} \\ & + \text{permutations}\big\} C_{2n}(g). \end{aligned} \tag{10.93}$$

For the spin-dependent structure function $G_1$ we have

$$\begin{aligned} C_3^{\mu\nu\mu_1\cdots\mu_n}(q) = \big\{ & \varepsilon^{\mu\nu\mu_1\alpha} S[q_\alpha q^{\mu_2} \cdots q^{\mu_n}](-q^2)^{(2-d-n)/2} \\ & + \text{permutations}\big\} C_{3n}(g). \end{aligned} \tag{10.94}$$

**(c)** Combining the results in (a) and (b), we have

$$\begin{aligned} & C_1^{\mu\nu\mu_1\cdots\mu_n}(q)\langle p, s | O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0) | p, s\rangle \\ & = -g^{\mu\nu} S[q^{\mu_1} q^{\mu_2} \cdots q^{\mu_n}](-q^2)^{(2-d-n)/2} M^{d-n-2} S[p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n}] \alpha_{nV} C_{1n} \\ & = -g^{\mu\nu}\left[\left(\frac{p\cdot q}{-q^2}\right)^n \left(\frac{-q^2}{M^2}\right)^{(2-d+n)/2} + \text{trace terms}\right] \alpha_{nV} C_{1n} \\ & = -g^{\mu\nu}\left[\left(\frac{1}{x}\right)^n \left(\frac{-q^2}{M^2}\right)^{(2-d+n)/2} + \cdots\right] \alpha_{nV} C_{1n}. \end{aligned} \tag{10.95}$$

Thus for twist-2 operator (recall twist is difference of spin and dimension, $d-n$), we have

$$W_1(q^2,\nu) = F_1(x,q^2) = \sum_n \left(\frac{1}{x}\right)^n \alpha_{nV} C_{1n}. \tag{10.96}$$

Similarly,

$$\begin{aligned} &C_2^{\mu\nu\mu_1\cdots\mu_n}(q)\langle p,s|O_{d,V}^{\mu_1\mu_2\cdots\mu_n}(0)|p,s\rangle \\ &= \frac{p^\mu p^\nu}{M^2}\left(\frac{1}{x}\right)^{n-2}\left(\frac{-q^2}{M^2}\right)^{(-d+n)/2} \alpha_{nV} C_{2n} \end{aligned}$$

## 10.5 Calculating Wilson coefficients

Since the Wilson coefficients are independent of processes, we can choose some simple external physical states, e.g. free quarks, to calculate these c-number coefficients.

The quark Compton scattering to lowest order in $\alpha_s$ is given by the diagrams in Fig. 10.3.

**(a)** From these diagrams, compute for massless quarks the amplitude

$$M_{\mu\nu} = \langle p,s|t_{\mu\nu}(q)|p,s\rangle \tag{10.97}$$

where

$$t_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} T(J_\mu(x)J_\nu(0)). \tag{10.98}$$

**(b)** For the operator-product expansion in the form

$$t_{\mu\nu}(q) \sim \sum_i C_i^{\mu\nu}(q) O^i(0), \tag{10.99}$$

there are two sets of flavour-singlet twist-2 operators,

$$\begin{aligned} O_{V,S}^{\mu_1\mu_2\cdots\mu_n}(x) &= \frac{1}{2}\frac{i^{n-1}}{n!}\left\{\bar{q}(x)\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_n}q(x) + \text{permutations}\right\} \\ O_{A,S}^{\mu_1\mu_2\cdots\mu_n}(x) &= \frac{1}{2}\frac{i^{n-1}}{n!}\left\{\bar{q}(x)\gamma^{\mu_1}\gamma_5 D^{\mu_2}\cdots D^{\mu_n}q(x) + \text{permutations}\right\}. \end{aligned}$$

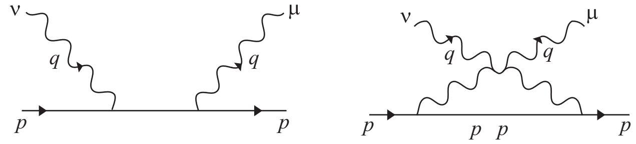



FIG. 10.3. Quark Compton scattering.

Show that the matrix elements of these operators between quark states are given, to lowest order in $\alpha_s$, by

$$\langle q(p,s)|O_{V,S}^{\mu_1\mu_2\cdots\mu_n}|q(p,s)\rangle = (p^{\mu_1}\cdots p^{\mu_n})$$
$$\langle q(p,s)|O_{A,S}^{\mu_1\mu_2\cdots\mu_n}|q(p,s)\rangle = h(p^{\mu_1}\cdots p^{\mu_n}) \qquad (10.100)$$

where $h$ is the helicity of the quark state $q(p,s)$ and is related to the polarization by

$$\bar{u}(p,s)\gamma_\mu\gamma_5 u(p,s) = s_\mu = 2hp_\mu. \qquad (10.101)$$

**(c)** If we write the operator-product expansion as

$$t_{\mu\nu}(q) = \Bigg\{ \Bigg[ -g^{\mu\nu}(q^{\mu_1}\cdots q^{\mu_n})\frac{1}{(-q^2)^n}C_{V,1}^{(ns)}$$
$$+(g^{\mu\mu_1}g^{\nu\mu_2}q^{\mu_3}\cdots q^{\mu_n})\frac{1}{(-q^2)^{n-1}}C_{V,2}^{(ns)} + \text{permutations}\Bigg] O_{V,S}^{\mu_1\mu_2\cdots\mu_n}$$
$$+\Bigg[\varepsilon^{\mu\nu\mu_1\alpha}q_\alpha q^{\mu_2}\cdots q^{\mu_n}\frac{1}{(-q^2)^n}C_A^{(ns)} + \text{permutations}\Bigg] O_{A,S}^{\mu_1\mu_2\cdots\mu_n}\Bigg\},$$

compute the Wilson coefficients to lowest order in $\alpha_s$ (i.e. in a free field theory).

### Solution to Problem 10.5

**(a)** From the Feynman diagrams in Fig. 10.3 we can write the amplitude

$$M_{\mu\nu} = i\bar{u}(p,s)\gamma_\mu\frac{i(\not{p}+\not{q})}{(p+q)^2}\gamma_\nu u(p,s) + i\bar{u}(p,s)\gamma_\nu\frac{i(\not{p}-\not{q})}{(p-q)^2}\gamma_\mu u(p,s). \qquad (10.102)$$

We now want to express this in terms of $q^2$ and $\omega = 2p\cdot q/(-q^2) = \frac{1}{x}$. We can expand the denominator as follows:

$$\frac{1}{(p+q)^2} = \frac{1}{2p\cdot q + q^2} = \frac{1}{q^2(1-\omega)} = \frac{1}{q^2}\sum_{n=0}^{\infty}\omega^n. \qquad (10.103)$$

Similarly,

$$\frac{1}{(p-q)^2} = \frac{1}{q^2}\sum_{n=0}^{\infty}(-1)^n\omega^n. \qquad (10.104)$$

In the first term we have

$$\bar{u}(p,s)\gamma_\mu(\not{p}+\not{q})\gamma_\nu u(p,s) = \bar{u}(p,s)[\gamma_\mu\not{q}\gamma_\nu + 2\gamma_\mu p_\nu]u(p,s). \qquad (10.105)$$

Using the identity CL-eqn (A.17), we get

$$\gamma_\mu\not{q}\gamma_\nu = q^\alpha(g_{\mu\alpha}\gamma_\nu + g_{\nu\alpha}\gamma_\mu - g_{\mu\nu}\gamma_\alpha + i\varepsilon_{\mu\nu\alpha\lambda}\gamma^\lambda\gamma_5). \qquad (10.106)$$

From the properties of Dirac spinors, we get

$$\bar{u}(p,s)\gamma_\mu u(p,s) = 2p_\mu, \qquad \bar{u}(p,s)\gamma_\mu\gamma_5 u(p,s) = s_\mu = 2p_\mu h, \tag{10.107}$$

where $s_\mu$ and $h$ are the polarization and helicity of the quark state. Then

$$\begin{aligned}\bar{u}(p,s)\gamma_\mu(\not{p}+\not{q})\gamma_\nu u(p,s) &= 4p_\mu p_\nu + 2p_\mu q_\nu + 2q_\mu p_\nu - 2g_{\mu\nu}(p\cdot q)\\ &\quad + 2i\varepsilon_{\mu\nu\alpha\lambda}q^\alpha p^\lambda h\end{aligned} \tag{10.108}$$

and the first term in $M_{\mu\nu}$ is

$$M^{(1)}_{\mu\nu} = \frac{-2}{q^2}\sum_{n=0}^{\infty}\omega^n\left[2p_\mu p_\nu + p_\mu q_\nu + q_\mu p_\nu - g_{\mu\nu}(p\cdot q) + i\varepsilon_{\mu\nu\alpha\lambda}q^\alpha p^\lambda h\right]. \tag{10.109}$$

To obtain the second term from $M^{(1)}_{\mu\nu}$ by the substitution $\mu \longleftrightarrow \nu,\ q \to -q$,

$$M^{(2)}_{\mu\nu} = \frac{-2}{q^2}\sum_{n=0}^{\infty}(-1)^n\omega^n\left[2p_\mu p_\nu - p_\mu q_\nu - q_\mu p_\nu + g_{\mu\nu}(p\cdot q) + i\varepsilon_{\mu\nu\alpha\lambda}q^\alpha p^\lambda h\right]. \tag{10.110}$$

The total amplitude is then

$$\begin{aligned}M_{\mu\nu} = \frac{-2}{q^2}\Bigg\{&2p_\mu p_\nu\sum_{n=0}^{\infty}[1+(-1)^n]\omega^n + g_{\mu\nu}(p\cdot q)\sum_{n=0}^{\infty}[1-(-1)^n]\omega^n\\ &+ i\varepsilon_{\mu\nu\alpha\lambda}q^\alpha p^\lambda\sum_{n=0}^{\infty}[1+(-1)^n]\omega^n + \cdots\Bigg\}.\end{aligned} \tag{10.111}$$

**(b)** Consider the simplest case $n = 1$, where we have the operator $O^\mu_{VS} = \frac{1}{2}\bar{q}\gamma^\mu q$. To the zeroth order $\alpha_s^0$, the free field theory limit, we can expand $q(x)$ as

$$q(x) = \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}}[b(p,s)e^{-ip\cdot x}u(p,s) + d^\dagger(p,s)e^{ip\cdot x}v(p,s)]. \tag{10.112}$$

It is then easy to see that

$$\begin{aligned}\langle q(p,s)|O^\mu_{V,S}|q(p,s)\rangle &= \tfrac{1}{2}\langle q(p,s)|\bar{q}\gamma^\mu q|q(p,s)\rangle\\ &= \tfrac{1}{2}\bar{u}(p,s)\gamma^\mu u(p,s) = p^\mu.\end{aligned} \tag{10.113}$$

Also to order $\alpha_s^0$, the covariant derivative $D_\mu$ is the same as the usual derivative $\partial_\mu$ and each gives a factor of $p_\mu$. Thus we get

$$\langle q(p,s)|O^{\mu_1\mu_2\cdots\mu_n}_{V,S}|q(p,s)\rangle = (p^{\mu_1}\cdots p^{\mu_n}). \tag{10.114}$$

For the axial vector current it is easy to see that

$$\tfrac{1}{2}\langle q(p,s)|\bar{q}\gamma^\mu\gamma_5 q|q(p,s)\rangle = \tfrac{1}{2}\bar{u}(p,s)\gamma^\mu\gamma_5 u(p,s) = hp^\mu \tag{10.115}$$

and

$$\langle q(p,s)|O^{\mu_1\mu_2\cdots\mu_n}_{A,S}|q(p,s)\rangle = h(p^{\mu_1}\cdots p^{\mu_n}). \tag{10.116}$$

**(c)** Taking the quark matrix element of $t_{\mu\nu}(q)$, we get

$$
\begin{aligned}
&\langle q(p,s)|t_{\mu\nu}(q)|q(p,s)\rangle \\
&\quad = \Bigg[ -g_{\mu\nu}(q^{\mu_1}q^{\mu_2}\cdots q^{\mu_n})\frac{1}{(-q^2)^n}C_{V,1}^{(ns)} \\
&\qquad + g_{\mu\mu_1}g_{\nu\mu_2}q^{\mu_3}\cdots q^{\mu_n}\frac{1}{(-q^2)^{n-1}}C_{V,2}^{(ns)} + \text{permutations}\Bigg] p_{\mu_1}p_{\mu_2}\cdots p_{\mu_n} \\
&\qquad + \Bigg[\varepsilon_{\mu\nu\mu_1\alpha}(q^{\alpha}q^{\mu_2}\cdots q^{\mu_n})\frac{1}{(-q^2)^n}C_A^{(ns)} + \text{permutations}\Bigg] hp_{\mu_1}p_{\mu_2}\cdots p_{\mu_n} \\
&\quad = -\,g_{\mu\nu}C_{V,1}^{(ns)}\left(\frac{1}{x}\right)^n + p_\mu p_\nu\left(\frac{1}{x}\right)^{n-2}\frac{1}{(-q^2)}C_{V,2}^{(ns)} \\
&\qquad + h\varepsilon_{\mu\nu\alpha\beta}q^{\alpha}p^{\beta}\left(\frac{1}{x}\right)^{n-1}\frac{1}{(-q^2)}C_A^{(ns)}.
\end{aligned}
\tag{10.117}
$$

Comparing this with the Compton amplitude given in (a), we see that

$$
\begin{aligned}
C_{V,1}^{(ns)} &= -2[1-(-1)^n], \qquad C_{V,2}^{(ns)} = -2[1+(-1)^n], \\
C_A^{(ns)} &= -2[1+(-1)^n].
\end{aligned}
\tag{10.118}
$$

# 11 Electroweak theory

## 11.1 Chiral spinors and helicity states

The Dirac spinor in momentum space can be written as,

$$u(p, \pm) = \sqrt{2m} \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \end{pmatrix} \chi_\pm \tag{11.1}$$

where $(\boldsymbol{\sigma} \cdot \hat{\mathbf{p}})\chi_\pm = \pm\chi_\pm$ with $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$. Show that the left-handed and right-handed spinors given by

$$u_L(p) = \tfrac{1}{2}(1 - \gamma_5)u(p, -), \qquad u_R(p) = \tfrac{1}{2}(1 + \gamma_5)u(p, +) \tag{11.2}$$

are eigenstates of the helicity operator $\lambda = \mathbf{s} \cdot \hat{\mathbf{p}}$ in the massless limit, where the spin operator is of the form

$$\mathbf{s} = \tfrac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{11.3}$$

Note that the same calculation should also show that the other two combinations

$$\tfrac{1}{2}(1 + \gamma_5)u(p, -), \qquad \tfrac{1}{2}(1 - \gamma_5)u(p, +) \tag{11.4}$$

are identically zero in the same limit.

**Solution to Problem 11.1**

In the standard representation, we have

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11.5}$$

Thus

$$\begin{aligned} u_L(p) = \tfrac{1}{2}(1 - \gamma_5)u(p, -) &= \tfrac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \end{pmatrix} \chi_- \\ &= \tfrac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \dfrac{-p}{E} \end{pmatrix} \chi_- = \tfrac{1}{2} \left( \dfrac{E + p}{E} \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \chi_- \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \chi_- \end{aligned} \tag{11.6}$$

where we have used $E = p$ for the massless particle. Similarly,

$$u_R(p) = \tfrac{1}{2}(1+\gamma_5)u(p,+) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \chi_+. \tag{11.7}$$

Then

$$\begin{aligned} \lambda u_L(p) &= \tfrac{1}{2}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\chi_- \\ &= \tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\chi_- = -\tfrac{1}{2}u_L(p). \end{aligned} \tag{11.8}$$

Similarly, we have

$$\lambda u_R(p) = \tfrac{1}{2}u_R(p). \tag{11.9}$$

## 11.2 The polarization vector for a fermion

For a particle described by a spinor $u(p,\lambda)$, we can define the polarization four-vector $s_\mu(p,\lambda)$ as

$$s_\mu(p,\lambda) = \frac{1}{2m}\bar{u}(p,\lambda)\gamma_\mu\gamma_5 u(p,\lambda). \tag{11.10}$$

**(a)** Show that

$$s\cdot p = 0. \tag{11.11}$$

**(b)** Calculate $s_\mu$ for the particle at rest ($\mathbf{p} = 0$), with

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{11.12}$$

**(c)** Show that

$$s^2 = -1. \tag{11.13}$$

**(d)** Suppose for a particle at rest the polarization vector is given by

$$s^\mu = (0, \boldsymbol{\eta}) \qquad \text{with} \quad \boldsymbol{\eta}^2 = 1. \tag{11.14}$$

Show that in the frame where the particle moves with momentum $\mathbf{p}$, the spin vector $s^\mu$ is given by

$$s^0 = \frac{\boldsymbol{\eta}\cdot\mathbf{p}}{m}, \qquad \mathbf{s} = \boldsymbol{\eta} + \frac{\mathbf{p}(\boldsymbol{\eta}\cdot\mathbf{p})}{(E+m)m}. \tag{11.15}$$

**Solution to Problem 11.2**

**(a)** Through a simple application of the Dirac equation, we have

$$s \cdot p = \frac{1}{2m}\bar{u}(p,\lambda)\not{p}\,\gamma_5 u(p,\lambda) = \frac{1}{2}\bar{u}(p,\lambda)\gamma_5 u(p,\lambda) \tag{11.16}$$

or, alternatively,

$$s \cdot p = \frac{1}{2m}\bar{u}(p,\lambda)\gamma_5(-\not{p})u(p,\lambda) = -\frac{1}{2}\bar{u}(p,\lambda)\gamma_5 u(p,\lambda). \tag{11.17}$$

Thus $s \cdot p = 0$.

**(b)** For a particle at rest, where we have $u(p,\lambda) = \sqrt{2m}\begin{pmatrix}1\\0\end{pmatrix}\chi_\lambda$, $p^\mu = (m,0,0,0)$, and $s \cdot p = 0$, we get

$$s_0 = 0 \tag{11.18}$$

and

$$\begin{aligned}\mathbf{s} &= \frac{1}{2m}\bar{u}(0,\lambda)\boldsymbol{\gamma}\gamma_5 u(0,\lambda)\\ &= \chi_\lambda^\dagger(1,0)\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\chi_\lambda = \chi_\lambda^\dagger\boldsymbol{\sigma}\chi_\lambda.\end{aligned} \tag{11.19}$$

Thus $s_1 = s_2 = 0$ and

$$s_3 = \begin{cases}1 & \text{for } \chi_+\\ -1 & \text{for } \chi_-\end{cases}. \tag{11.20}$$

This means $\mathbf{s}$ is in the direction of the spin. In this simple frame we have

$$s^\mu = (0,0,0,\pm 1), \qquad s^2 = -1. \tag{11.21}$$

**(c)** The spin vector

$$s_\mu(p,\lambda) = \frac{1}{2m}\bar{u}(p,\lambda)\gamma_\mu\gamma_5 u(p,\lambda) \tag{11.22}$$

transforms as a four-vector under Lorentz transformations. Thus $s^2 = s^\mu s_\mu$ is a Lorentz scalar and $s^2 = -1$ in all frames.

**(d)** Since $\boldsymbol{\eta}$ and $\mathbf{p}$ are the only vectors in the problem, we can write

$$\mathbf{s} = a\boldsymbol{\eta} + b\mathbf{p}, \qquad a \text{ and } b \text{ are constants.} \tag{11.23}$$

Since we are given $\mathbf{s} = \boldsymbol{\eta}$ when the particle is at rest at $\mathbf{p} = 0$, we must have $a = 1$. From $s \cdot p = 0$, we get

$$s_0 = \frac{1}{E}(\boldsymbol{\eta} + b\mathbf{p}) \cdot \mathbf{p} = \frac{1}{E}(\boldsymbol{\eta} \cdot \mathbf{p} + bp^2) \tag{11.24}$$

and the condition $s^2 = -1$ can now be written as

$$s_0^2 - \mathbf{s}^2 = s_0^2 - (\boldsymbol{\eta} + b\mathbf{p})^2 = -1. \tag{11.25}$$

Using eqn (11.24), this leads to

$$\frac{1}{E^2}(\boldsymbol{\eta} \cdot \mathbf{p} + bp^2)^2 = (\boldsymbol{\eta} + b\mathbf{p})^2 - 1 \tag{11.26}$$

or

$$b^2(E^2 - m^2)m^2 + 2b(\boldsymbol{\eta} \cdot \mathbf{p})m^2 - (\boldsymbol{\eta} \cdot \mathbf{p})^2 = 0, \tag{11.27}$$

or

$$[m(E-m)b + (\boldsymbol{\eta} \cdot \mathbf{p})][m(E+m)b - (\boldsymbol{\eta} \cdot \mathbf{p})] = 0. \tag{11.28}$$

This gives the solution

$$b = \frac{(\boldsymbol{\eta} \cdot \mathbf{p})}{m(E+m)}. \tag{11.29}$$

(The other solution does not go to zero as $\mathbf{p} \to 0$.) Thus we have

$$\mathbf{s} = \boldsymbol{\eta} + \frac{\mathbf{p}(\boldsymbol{\eta} \cdot \mathbf{p})}{(E+m)m}. \tag{11.30}$$

## 11.3 The pion decay rate and $f_\pi$

The decay $\pi^+ \to \mu^+ + \nu_\mu$ is described by the effective Lagrangian for the four-fermion interaction

$$\mathcal{L}_{wk} = -\frac{G_F}{\sqrt{2}} \cos\theta_c \left[\bar{u}\gamma^\mu(1-\gamma_5)d\right]\left[\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu\right]. \tag{11.31}$$

**(a)** Show that

$$\langle 0|\bar{u}\gamma_\mu d|\pi^+(p)\rangle = 0 \tag{11.32}$$

because parity is conserved in the strong interaction.

**(b)** Show that the general form of the axial vector current is given by

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi^+(p)\rangle = i\sqrt{2} f_\pi p_\mu \tag{11.33}$$

where $f_\pi$ is the pion decay constant.

**(c)** Calculate the decay rate for $\pi^+ \to \mu^+ + \nu_\mu$ and use the measured lifetime $\tau_\pi = 2.6 \times 10^{-8}$ s to determine the constant $f_\pi$.

**(d)** Show that as a consequence of the V−A theory, the amplitude for the decay $\pi^+ \to \mu^+\nu_\mu$ is proportional to $m_\mu$, and to $m_e$ for $\pi^+ \to e^+\nu_e$.

**Solution to Problem 11.3**

**(a)** The parity conservation of the strong interaction implies that

$$\mathcal{P}(\bar{u}\gamma_\mu d)\mathcal{P}^{-1} = \bar{u}\gamma^\mu d, \qquad \mathcal{P}|\pi^+(\mathbf{p})\rangle = -|\pi^+(-\mathbf{p})\rangle. \tag{11.34}$$

Thus the matrix element

$$\begin{aligned}\langle 0|\bar{u}\gamma_\mu d|\pi^+(\mathbf{p})\rangle &= \langle 0|\mathcal{P}^{-1}\mathcal{P}(\bar{u}\gamma_\mu d)\mathcal{P}^{-1}\mathcal{P}|\pi^+(\mathbf{p})\rangle \\ &= -\langle 0|\bar{u}\gamma^\mu d|\pi^+(-\mathbf{p})\rangle.\end{aligned} \tag{11.35}$$

This means that for the time component, we have

$$\langle 0|\bar{u}\gamma_0 d|\pi^+(\mathbf{p})\rangle = -\langle 0|\bar{u}\gamma_0 d|\pi^+(-\mathbf{p})\rangle \tag{11.36}$$

or

$$\langle 0|\bar{u}\gamma_0 d|\pi^+(\mathbf{p}=0)\rangle = 0. \tag{11.37}$$

For the spatial components, we get

$$\langle 0|\bar{u}\boldsymbol{\gamma} d|\pi^+(\mathbf{p})\rangle = \langle 0|\bar{u}\boldsymbol{\gamma} d|\pi^+(-\mathbf{p})\rangle. \tag{11.38}$$

This matrix element is a three-vector under rotation and the only three-vector this can depend on is $\mathbf{p}$, which changes sign under parity. Thus

$$\langle 0|\bar{u}\boldsymbol{\gamma} d|\pi^+(\mathbf{p})\rangle = 0. \tag{11.39}$$

In essence, this argument simply says that since $\pi^+$ is a pseudoscalar, the matrix element of vector current $\langle 0|\bar{u}\gamma_\mu d|\pi^+(\mathbf{p})\rangle$ is an axial-vector while the only vector it can depend on, $p_\mu$, is a polar vector. Therefore, this matrix element must vanish.

**(b)** Using the same argument, we see that $\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi^+(\mathbf{p})\rangle$ is a polar vector and has to be proportional to $p_\mu$:

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi^+(p)\rangle = i\sqrt{2}f_\pi p_\mu. \tag{11.40}$$

**(c)** The matrix element for the decay is of the form

$$\begin{aligned}\mathcal{M} &= -i\frac{G_F}{\sqrt{2}}\cos\theta_c\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi^+(p)\rangle\bar{v}(k_2)\gamma^\mu(1-\gamma_5)u(k_1) \\ &= \frac{G_F f_\pi}{\sqrt{2}}\cos\theta_c\, p_\mu\bar{v}(k_2)\gamma^\mu(1-\gamma_5)u(k_1) \\ &= \frac{G_F f_\pi}{\sqrt{2}}\cos\theta_c\, m_\mu\bar{v}(k_2)(1-\gamma_5)u(k_1)\end{aligned} \tag{11.41}$$

where $p = k_1 + k_2$, with $k_2$ being the momentum of the muon. Note that this matrix element is proportional to the lepton mass $m_\mu$. The decay rate is then given by

$$\Gamma = \frac{1}{2m_\pi}\int (2\pi)^4\delta^4(p-k_1-k_2)\frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}\sum_{spin}|\mathcal{M}|^2 \qquad (11.42)$$

and

$$\begin{aligned}\sum_{spin}|\mathcal{M}|^2 &= G_F^2 f_\pi^2\cos^2\theta_c m_\mu^2 Tr[(\not k_2 - m_\mu)(1-\gamma_5)\not k_1(1+\gamma_5)]\\ &= G_F^2 f_\pi^2\cos^2\theta_c m_\mu^2 8(k_1\cdot k_2)\\ &= 4G_F^2 f_\pi^2\cos^2\theta_c m_\mu^2\left(m_\pi^2 - m_\mu^2\right) \qquad (11.43)\end{aligned}$$

where we have use the relation

$$2(k_1\cdot k_2) = (k_1+k_2)^2 - k_1^2 - k_2^2 = m_\pi^2 - m_\mu^2. \qquad (11.44)$$

The phase space can be calculated easily to yield

$$\begin{aligned}\rho &= \int (2\pi)^4\delta^4(p-k_1-k_2)\frac{d^3k_1}{(2\pi)^3\,2E_1}\frac{d^3k_2}{(2\pi)^3\,2E_2}\\ &= \frac{1}{(2\pi)^2}\int \delta(m_\pi - E_1 - E_2)\frac{d^3k_1}{(2\pi)^3 4E_1E_2}. \qquad (11.45)\end{aligned}$$

For the pion at rest

$$p_0 = m_\pi, \qquad \mathbf{k}_1 + \mathbf{k}_2 = 0, \qquad d^3k_1 = 4\pi k_1^2 dk_1 = 4\pi E_1^2 dE_1, \qquad (11.46)$$

$$E_2 = (m_\mu^2 + \mathbf{k}_2^2)^{1/2} = \left(m_\mu^2 + E_1^2\right)^{1/2}, \qquad (11.47)$$

and

$$\rho = \frac{1}{\pi}\int \delta\left[m_\pi - E_1 - \left(m_\mu^2 + E_1^2\right)^{1/2}\right]\frac{E_1 dE_1}{4E_2}. \qquad (11.48)$$

Let $x = E_1 + (m_\mu^2 + E_1^2)^{1/2}$, then

$$\begin{aligned}dx &= dE_1 + \frac{E_1 dE_1}{\left(m_\mu^2 + E_1^2\right)^{1/2}}\\ &= \frac{dE_1}{\left(m_\mu^2 + E_1^2\right)^{1/2}}\left[E_1 + \left(m_\mu^2 + E_1^2\right)^{1/2}\right] = \frac{x\,dE_1}{E_2} \qquad (11.49)\end{aligned}$$

and

$$\rho = \frac{1}{4\pi}\int \delta(m_\pi - x)E_1\frac{dx}{x} = \frac{1}{4\pi}\frac{E_1}{m_\pi}. \qquad (11.50)$$

For $x = m_\pi$, we get $m_\pi = E_1 + (m_\mu^2 + E_1^2)^{1/2}$ or $E_1 = (m_\pi^2 - m_\mu^2)/2m_\pi$. The phase space is then

$$\rho = \frac{1}{4\pi}\frac{m_\pi^2 - m_\mu^2}{2m_\pi}. \tag{11.51}$$

The decay rate is

$$\begin{aligned}\Gamma &= \frac{1}{2m_\pi}\frac{\left(m_\pi^2 - m_\mu^2\right)}{8\pi m_\pi} 4 f_\pi^2 G_F^2 \cos^2\theta_c m_\mu^2 \left(m_\pi^2 - m_\mu^2\right) \\ &= \frac{G_F^2}{4\pi} f_\pi^2 m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \cos^2\theta_c = \frac{1}{\tau_\pi}.\end{aligned} \tag{11.52}$$

Substitute in the pion lifetime, the Fermi constant, and the Cabibbo angle, etc., and we can deduce $f_\pi = 0.66\, m_\pi \simeq 90\,\text{MeV}$.

**(d)** In the V–A theory $\nu_\mu$ is left-handed and $\mu^+$ is right-handed. In the limit $m_\mu = 0$, $\mu^+$ has helicity $\frac{1}{2}$. Thus in the rest frame of $\pi^+$, $\mu^+$ and $\nu_\mu$ come out back-to-back and the total spin along the direction of $\mu^+$is +1 (see Fig. 11.1). However, $\pi^+$ has spin zero. Thus this decay is forbidden in the limit $m_\mu = 0$ and the decay can proceed only if $m_\mu \neq 0$, see eqn (11.41), in which case right-handed $\mu^+$ is not a pure helicity state.

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left[\frac{\left(1 - (m_e^2/m_\pi^2\right)^2}{\left(1 - (m_\mu^2/m_\pi^2)\right)^2}\right] = 1.23 \times 10^{-4}. \tag{11.53}$$

Thus, pions decay predominantly into muon leptons rather than electron leptons.

**Remark.** If we use the same analysis for the charm meson decays, the results are very similar:

$$\frac{\Gamma(F^+ \to \tau^+\nu_\tau)}{\Gamma(F^+ \to \mu^+\nu_\mu)} = \left(\frac{m_\tau}{m_\mu}\right)^2 \left[\frac{\left(1 - (m_\tau^2/m_F^2)\right)^2}{\left(1 - (m_\mu^2/m_F^2)\right)^2}\right] \simeq 17, \tag{11.54}$$

$$\frac{\Gamma(D^+ \to \tau^+\nu_\tau)}{\Gamma(D^+ \to \mu^+\nu_\mu)} = \left(\frac{m_\tau}{m_\mu}\right)^2 \left[\frac{\left(1 - (m_\tau^2/m_D^2)\right)^2}{\left(1 - (m_\mu^2/m_D^2)\right)^2}\right] \simeq 2.5. \tag{11.55}$$

Note that leptonic decays of $D$s are suppressed by $\sin^2\theta_c$.

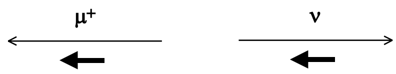



FIG. 11.1. $\pi \to \mu^+\nu$ decay would be forbidden in the $m_\mu = 0$ limit.

## 11.4 Uniqueness of the standard model scalar potential

The usual $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ scalar potential for the standard model is of the form

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \tag{11.56}$$

with

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{11.57}$$

**(a)** In principle, one can also have an $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ quartic invariant of the form

$$V_1(\phi) = \lambda_1 (\phi^\dagger \boldsymbol{\tau} \phi) \cdot (\phi^\dagger \boldsymbol{\tau} \phi). \tag{11.58}$$

Show that this quartic term can be reduced to that in $V(\phi)$ of eqn (11.56).

**(b)** Show that another quartic term

$$V_2(\phi) = \lambda_2 \sum_{a,b} \left(\phi^\dagger \tau^a \tau^b \phi\right) \left(\phi^\dagger \tau^a \tau^b \phi\right) \tag{11.59}$$

is also reducible to that in $V(\phi)$ of eqn (11.56).

### Solution to Problem 11.4

**(a)** Writing out the components

$$(\phi^\dagger \boldsymbol{\tau} \phi) \cdot (\phi^\dagger \boldsymbol{\tau} \phi) = \sum_{i,j,k,l} \left(\phi_i^* \phi_j\right) \left(\phi_k^* \phi_l\right) \sum_a (\tau^a)_{ij} (\tau^a)_{kl} \tag{11.60}$$

and using the identity

$$\sum_a (\tau^a)_{ij} (\tau^a)_{kl} = 2 \left( \delta_{jk} \delta_{il} - \frac{1}{2} \delta_{ij} \delta_{kl} \right), \tag{11.61}$$

we obtain $(\phi^\dagger \boldsymbol{\tau} \phi) \cdot (\phi^\dagger \boldsymbol{\tau} \phi) = (\phi^\dagger \phi)^2$.

**(b)** We can use the identity in eqn (11.61) to derive

$$\begin{aligned}
\sum_{a,b} (\tau^a \tau^b)_{ij} (\tau^a \tau^b)_{kl} &= \sum_{a,b} (\tau^a)_{im} (\tau^b)_{ml} (\tau^a)_{kn} (\tau^b)_{nl} \\
&= 2 \left( \delta_{mk} \delta_{in} - \frac{1}{2} \delta_{im} \delta_{kn} \right) 2 \left( \delta_{jn} \delta_{mil} - \frac{1}{2} \delta_{mj} \delta_{nl} \right) \\
&= 4 \left( \delta_{kl} \delta_{ij} - \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2} \delta_{jk} \delta_{il} + \frac{1}{4} \delta_{ij} \delta_{kl} \right) \\
&= (5 \delta_{kl} \delta_{ij} - 4 \delta_{il} \delta_{jk}).
\end{aligned} \tag{11.62}$$

Thus we have

$$\sum_{a,b} (\phi^\dagger \tau^a \tau^b \phi)(\phi^\dagger \tau^a \tau^b \phi) = (\phi^\dagger \phi)^2. \tag{11.63}$$

**Remark.** Clearly we can generalize this result to the more general case of vector representation in SU($n$), e.g.

$$(\phi^\dagger \boldsymbol{\lambda}\phi)(\phi^\dagger \boldsymbol{\lambda}\phi) = 2\left[(\phi^\dagger\phi)^2 - \frac{1}{n}(\phi^\dagger\phi)^2\right] = \frac{2(n-1)}{n}(\phi^\dagger\phi)^2 \qquad (11.64)$$

by using the identity

$$\sum_a (\lambda^a)_{ij}(\lambda^a)_{kl} = 2\left(\delta_{jk}\delta_{il} - \frac{1}{n}\delta_{ij}\delta_{kl}\right). \qquad (11.65)$$

## 11.5 Electromagnetic and gauge couplings

In the more general case, the interaction of neutral gauge bosons can be written in the form

$$\mathcal{L}_N = \sum_{i=1}^{n} g_i J^{i\mu} A^i_\mu \qquad (11.66)$$

where $g_1, \ldots, g_n$ are gauge coupling constants, $J^{1\mu}, \ldots, J^{n\mu}$ are various neutral currents, and $A^i_\mu$ are the neutral gauge boson fields, which are gauge eigenstates. Suppose $A^i_\mu$ is written in terms of mass eigenstates as

$$A^i_\mu = \sum_{i=1}^{n} S_{ia} W^a_\mu \qquad (11.67)$$

where $S$ is an orthogonal matrix with property

$$\sum_{i=1}^{n} S_{ia}S_{ib} = \delta_{ab}, \qquad \sum_{a=1}^{n} S_{ia}S_{ja} = \delta_{ij}. \qquad (11.68)$$

**(a)** Show that the electric charge $e$ is related to the gauge couplings by

$$\frac{1}{e^2} = \sum_{j=1}^{n}\left(\frac{c_j}{g_j}\right)^2 \qquad (11.69)$$

where $c_j$s are the coefficients of $J^{i\mu}$ in the electromagnetic current: $J^{em}_\mu = \sum_j c_j J^j_\mu$, or in terms of the charge operator, $Q = \sum_j c_j Y^j$ with $Y^i = \int d^3x\, J^i_0$.

**(b)** Use the result in eqn (11.69) to derive, in the standard model, the relation

$$e = g \sin\theta_W. \qquad (11.70)$$

**Solution to Problem 11.5**

**(a)** The neutral current coupling can be rewritten as

$$\mathcal{L}_N = \sum_{i=1}^{n} g_i J^{i\mu} A^i_\mu = \sum_{i,a} g_i J^{i\mu} S_{ia} W^a_\mu = \sum_{a=1}^{n} e_a J^{a\mu} W^a_\mu \qquad (11.71)$$

where $e_a J^a_\mu = \sum_i g_i J^i_\mu S_{ia}$. For the case where $J^a_\mu = J^{em}_\mu$, the electromagnetic current, we get

$$eJ^{em}_\mu = \sum_{i=1}^{n} g_i J^i_\mu S_{iQ}. \tag{11.72}$$

Using $J^{em}_\mu = \sum_j c_j J^j_\mu$ we get

$$e\sum_j c_j J^j_\mu = \sum_{i=1}^{n} g_i J^i_\mu S_{iQ}. \tag{11.73}$$

Identifying the coefficient of $J^i_\mu$ (for a given $i$), we get

$$ec_i = g_i S_{iQ} \qquad \text{or} \qquad S_{iQ} = \frac{ec_i}{g_i}. \tag{11.74}$$

From the fact that $S$ is a orthogonal matrix, $\sum_i (S_{iQ})^2 = 1$, we get

$$e^2 \sum_{i=1}^{n} \left(\frac{c_i}{g_i}\right)^2 = 1 \qquad \text{or} \qquad \sum_{i=1}^{n} \left(\frac{c_i}{g_i}\right)^2 = \frac{1}{e^2}. \tag{11.75}$$

**(b)** For the specific case of the SU(2) $\times$ U(1) theory, we have

$$\mathcal{L}_N = gJ^{3\mu}A^3_\mu + \frac{g'}{2}J^{Y\mu}B_\mu. \tag{11.76}$$

Namely, $g_1 = g$ and $g_2 = \frac{g'}{2}$. We also have $c_1 = 1$ and $c_2 = \frac{1}{2}$, because

$$Q = T_3 + \frac{Y}{2} \qquad \text{or} \qquad J^{em}_\mu = J^3_\mu + \frac{1}{2}J^Y_\mu.$$

Then the relation in eqn (11.75) becomes

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}. \tag{11.77}$$

Using $g' = g\tan\theta_W$, we get

$$\frac{1}{e^2} = \frac{1}{g^2}(1 + \cot^2\theta_W) \qquad \text{or} \qquad e = g\sin\theta_W. \tag{11.78}$$

## 11.6 Fermion mass-matrix diagonalization

Suppose that the fermion mass matrix in the basis of left-handed and right-handed fields is hermitian,

$$\mathcal{L}_M = \bar{\psi}_{iL} M_{ij} \psi_{jR} + h.c. \qquad M^\dagger = M. \tag{11.79}$$

In general, the eigenvalues of $M$ obtained from a unitary transformation are not always positive:

$$UMU^\dagger = M_d = diag(m_1, m_2, \ldots, m_n) \tag{11.80}$$

where $m_i$ can be negative as well as positive.

**(a)** Show that one can choose an appropriate biunitary transformation to diagonalize $M$ so that all diagonal elements are non-negative.

**(b)** If the mass matrix is real, show that the matrices in the biunitary transformation can be chosen to be orthogonal matrices.

**Solution to Problem 11.6**

**(a)** For the cases where some of the $m_i$s are negative in the diagonal matrix $M_d = UMU^\dagger$, we can always find a diagonal matrix $S$, consisting of 1 or $-1$, such that the product $M_dS$ is a positive matrix:

$$\overline{M}_d = M_d S \geqslant 0. \tag{11.81}$$

Then

$$\overline{M}_d = M_d S = UMU^\dagger S = UMV^\dagger \tag{11.82}$$

with $U^\dagger S = V^\dagger$. Since both $U$ and $S$ are unitary, $V$ is also unitary. Then $\overline{M}_d$ is in the form of a biunitary transformation. In this way we can make all fermion masses non-negative. Clearly, this can also be done even when $M$ is not hermitian.

**(b)** If $M$ is real then $MM^\dagger$ is real and symmetric and can be diagonalized by orthogonal transformation:

$$S(MM^\dagger)S^T = M_d^2 = \begin{pmatrix} m_1^2 & & \\ & \ddots & \\ & & m_n^2 \end{pmatrix}, \quad m_i \geqslant 0. \tag{11.83}$$

Let us define

$$M_d = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \qquad \text{and} \qquad H = SM_dS^T, \tag{11.84}$$

then $H$ is real and symmetric. Define $T$ by $T \equiv H^{-1}M$, then

$$TT^T = H^{-1}MM^\dagger(H^{-1})^T = H^{-1}SM_d^2S^TH^{-1} = H^{-1}H^2H^{-1} = 1, \tag{11.85}$$

i.e. T is orthogonal. We have

$$M = HT = SM_dS^TT \qquad \text{or} \qquad M_d = (S^TMTS) = S^TMR \tag{11.86}$$

where $R = TS$, which is also an orthogonal matrix.

## 11.7 An example of calculable mixing angles

The properties of the mass matrix can be translated into relations between mass values and mixing angles. Here is an illustrative example of such a model. Consider a simple $2 \times 2$ hermitian fermion mass matrix of the form

$$M = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix}. \tag{11.87}$$

Show that the mixing angle $\theta$ which characterizes the $2 \times 2$ unitary matrix which diagonalizes $M$ is related to the mass eigenvalues by

$$\tan\theta = \sqrt{\frac{m_1}{m_2}}. \tag{11.88}$$

**Solution to Problem 11.7**

The mass matrix can be diagonalized by an orthogonal transformation

$$SMS^\dagger = M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \text{or} \qquad M = S^\dagger M_d S \tag{11.89}$$

with

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{11.90}$$

From $M_{11} = 0$, we get

$$S_{1i}^*(M_d)_{ij}S_{j1} = 0 \qquad \text{or} \qquad (\cos^2\theta)m_1 - (\sin^2\theta)m_2 = 0 \tag{11.91}$$

or

$$\tan\theta = \sqrt{\frac{m_1}{m_2}}. \tag{11.92}$$

**Remark.** Attempts to relate the Cabibbo angle to the strange and down quark masses have been carried out along such approaches.

## 11.8 Conservation of the $B - L$ quantum number

Show that if there were a set of scalars transforming as a doublet under the weak SU(2) symmetry and as a triplet under the colour $\mathrm{SU}(3)_C$: $h_\alpha^i (i = 1, 2, \alpha = 1, 2, 3)$, then both the baryon number $B$ and the lepton number $L$ are not conserved. However, the linear combination $B - L$ is conserved.

**Solution to Problem 11.8**

As mentioned in CL-p. 355, the presence of $h_\alpha^i$ will lead to Yukawa couplings of the form

$$\mathcal{L}_Y = f_{h_1}\bar{l}_{iL}h_\alpha^i q_R^\alpha + f_{h_2}\bar{q}_{i\alpha L}h_\beta^i q_{\gamma R}^c \varepsilon^{\alpha\beta\gamma}. \tag{11.93}$$

In order to conserve the quantum number $B$, the first term requires the assignment $B_1(h) = -\frac{1}{3}$, while second term requires $B_2(h) = \frac{2}{3}$. Thus the baryon number $B$

is not conserved. For the lepton number conservation we have $L_1(h) = -1$ and $L_2(h) = 0$, and the lepton number is not conserved. However, the combination $B - L$ has the values

$$B_1 - L_1 = \frac{2}{3} \qquad \text{and} \qquad B_2 - L_2 = \frac{2}{3} \tag{11.94}$$

and is conserved by these Yukawa couplings.

**Remark.** This simple example illustrates that the baryon number (or lepton number) conservation is an 'accidental symmetry' due to some special structure of the Higgs potential.

# 12 Electroweak phenomenology

## 12.1 Atomic parity violation

The weak neutral-current interaction mediated by the Z boson in an atom violates the parity conservation and will generate mixing between levels with opposite parities.

**(a)** Show that the parity-violating part of the interaction has the form

$$\mathcal{L}_N = \frac{g^2}{2M_w^2}\left(A_e^\mu V_\mu^q + V_e^\mu A_\mu^q\right) \tag{12.1}$$

where $A_e^\mu$ and $V_e^\mu$ are the axial and vector currents of the electron and $A_\mu^q$ and $V_\mu^q$ are the axial and vector currents of quarks.

**(b)** If we write $\mathcal{L}_N$ in the form

$$\mathcal{L}_N = \frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma_\mu\gamma_5 e\left(C_{1u}\bar{u}\gamma_\mu u + C_{1d}\bar{d}\gamma_\mu d\right) + \bar{e}\gamma_\mu e\left(C_{2u}\bar{u}\gamma_\mu\gamma_5 u + C_{2d}\bar{d}\gamma_\mu\gamma_5 d\right)\right] \tag{12.2}$$

calculate the coefficients $C_{iu}$ and $C_{id}$.

**(c)** Using the fact that the momentum transfer is small in the atomic processes, show that we can write the effective interaction in terms of the nucleon fields $(p, n)$ as

$$\mathcal{L}'_N = \frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma_\mu\gamma_5 e\left(C_{1p}\bar{p}\gamma_\mu p + C_{1n}\bar{n}\gamma_\mu n\right) + \bar{e}\gamma_\mu e\left(C_{2p}\bar{p}\gamma_\mu\gamma_5 p + C_{2n}\bar{n}\gamma_\mu\gamma_5 n\right)\right] \tag{12.3}$$

**(d)** Show that for the case of heavy atoms, the terms containing vector currents of the nucleons add coherently and are much larger than the axial vector terms. The interaction can be written as

$$\mathcal{L}'_N = \frac{G_F}{2\sqrt{2}} Q_{wk} e^\dagger \gamma_5 e \tag{12.4}$$

where $Q_{wk} = (1 - 4\sin^2\theta_W)Z - N$ is the weak charge of the nucleus with $Z$ protons and $N$ neutrons.

**Solution to Problem 12.1**

**(a)** The neutral current interaction is of the form

$$\mathcal{L}_N = \frac{g^2}{2M_W^2} J_\mu^N J^{N\mu} \tag{12.5}$$

where the neutral current $J_\mu^N$ contains lepton and quark currents, and each current has a vector and an axial vector part. Thus the parity-violating interaction due to the exchange of a Z boson between electrons and the $u$,$d$ quarks in the nucleus must result from the following V–A interference:

$$\mathcal{L}_N = \frac{g^2}{M_W^2}\left[A^{e\mu}V_\mu^q + V^{e\mu}A_\mu^q\right] \tag{12.6}$$

where $V_\mu^{e,q}$ and $A_\mu^{e,q}$ are the electron/quark vector and axial vector neutral currents, respectively.

**(b)** The neutral current having the general structure of $J_\mu^N \propto \left(T_3 - \sin^2\theta_W Q\right)$ for the electron, we have

$$\begin{aligned} J_\mu^N(e) &= \bar{e}_L\gamma_\mu\left(-\tfrac{1}{2} + \sin^2\theta_W\right)e_L + \bar{e}_R\gamma_\mu \sin^2\theta_W\ e_R \\ &= \left(-\tfrac{1}{4} + \sin^2\theta_W\right)\bar{e}\gamma_\mu\, e + \tfrac{1}{4}\bar{e}\gamma_\mu\,\gamma_5 e \end{aligned} \tag{12.7}$$

and thus

$$V_\mu^e = \left(-\tfrac{1}{4} + \sin^2\theta_W\right)\bar{e}\gamma_\mu\, e \qquad \text{and} \qquad A_\mu^e = \tfrac{1}{4}\bar{e}\gamma_\mu\,\gamma_5 e. \tag{12.8}$$

For the $u$ and $d$ quarks, we have

$$\begin{aligned} J_\mu^N(u) &= \bar{u}_L\gamma_\mu\left(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W\right)u_L - \tfrac{2}{3}\sin^2\theta_W\bar{u}_R\gamma_\mu\, u_R \\ &= \left(\tfrac{1}{4} - \tfrac{2}{3}\sin^2\theta_W\right)\bar{u}\gamma_\mu u - \tfrac{1}{4}\bar{u}\gamma_\mu\gamma_5 u \end{aligned} \tag{12.9}$$

and

$$\begin{aligned} J_\mu^N(d) &= \bar{d}_L\gamma_\mu\left(-\tfrac{1}{2} + \tfrac{1}{3}\sin^2\theta_W\right)d_L + \tfrac{1}{3}\sin^2\theta_W\bar{d}_R\gamma_\mu d_R \\ &= \left(-\tfrac{1}{4} + \tfrac{1}{3}\sin^2\theta_W\right)\bar{d}\gamma_\mu d + \tfrac{1}{4}\bar{d}\gamma_\mu\gamma_5 d. \end{aligned} \tag{12.10}$$

Then, using $g^2/8M_W^2 = G_F/\sqrt{2}$, we can write,

$$\begin{aligned} \mathcal{L}_N = \frac{G_F}{\sqrt{2}}\Big\{ & 2\bar{e}\gamma_\mu\gamma_5 e\left[\left(\tfrac{1}{4} - \tfrac{2}{3}\sin^2\theta_W\right)\bar{u}\gamma_\mu u + \left(-\tfrac{1}{4} + \tfrac{1}{3}\sin^2\theta_W\right)\bar{d}\gamma_\mu d\right] \\ & + \left(-\tfrac{1}{4} + \sin^2\theta_W\right)\bar{e}\gamma_\mu e 2\left(\bar{d}\gamma_\mu\gamma_5 d - \bar{u}\gamma_\mu\gamma_5 u\right)\Big\}. \end{aligned} \tag{12.11}$$

Reading out the coefficients, we get

$$\begin{aligned} C_{1u} &= 2\left(\tfrac{1}{4} - \tfrac{2}{3}\sin^2\theta_W\right), \quad C_{1d} = 2\left(-\tfrac{1}{4} + \tfrac{1}{3}\sin^2\theta_W\right), \\ C_{2u} &= -2\left(-\tfrac{1}{4} + \sin^2\theta_W\right), \quad C_{2d} = 2\left(-\tfrac{1}{4} + \sin^2\theta_W\right). \end{aligned} \tag{12.12}$$

**(c)** It is convenient to write $\mathcal{L}_N$ of eqn (**??**) in the form

$$\begin{aligned} \mathcal{L}_N &= \frac{G_F}{\sqrt{2}}\Big\{\bar{e}\gamma_\mu\gamma_5 e\left[\left(\tfrac{1}{2}\bar{u}\gamma_\mu u - \tfrac{1}{2}\bar{d}\gamma_\mu d\right) - 2\sin^2\theta_W\left(\tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d\right)\right] \\ &\quad + \left(-\tfrac{1}{4} + \sin^2\theta_W\right)4\bar{e}\gamma_\mu e\left(\tfrac{1}{2}\bar{u}\gamma_\mu\gamma_5 u - \tfrac{1}{2}\bar{d}\gamma_\mu\gamma_5 d\right)\Big\} \\ &= \frac{G_F}{\sqrt{2}}\Big\{\bar{e}\gamma_\mu\gamma_5 e\left(V_\mu^3 - 2\sin^2\theta_W J_\mu^{em}\right) + \left(-\tfrac{1}{4} + \sin^2\theta_W\right)4\bar{e}\gamma_\mu e A_\mu^3\Big\} \end{aligned} \tag{12.13}$$

where

$$V_\mu^3 = \tfrac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right), \quad A_\mu^3 = \left(\tfrac{1}{2}\bar{u}\gamma_\mu\gamma_5 u - \tfrac{1}{2}\bar{d}\gamma_\mu\gamma_5 d\right).$$
$$J_\mu^{em} = \left(\tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d\right). \tag{12.14}$$

The nucleon matrix elements of these operators are (in the limit of zero momentum transfer)

$$\begin{aligned}
\langle p|V_\mu^3|p\rangle &= \tfrac{1}{2}\bar{p}\gamma_\mu p, & \langle n|V_\mu^3|n\rangle &= -\tfrac{1}{2}\bar{n}\gamma_\mu n,\\
\langle p|A_\mu^3|p\rangle &= \tfrac{1}{2}g_A\bar{p}\gamma_\mu\gamma_5 p, & \langle n|A_\mu^3|n\rangle &= -\tfrac{1}{2}g_A\bar{n}\gamma_\mu\gamma_5 n,\\
\langle p|J_\mu^{em}|p\rangle &= \bar{p}\gamma_\mu p, & \langle n|J_\mu^{em}|n\rangle &= 0,
\end{aligned} \tag{12.15}$$

where $(p,\ n)$ are the proton and neutron spinors, respectively, and $g_A = -1.25$ is the usual axial vector coupling constant of the nucleon. In terms of nucleon fields, we can write

$$\begin{aligned}
\mathcal{L}_N = \frac{G_F}{\sqrt{2}}\Big\{&\bar{e}\gamma_\mu\gamma_5 e\left[\left(\tfrac{1}{2}\bar{p}\gamma_\mu p - \tfrac{1}{2}\bar{n}\gamma_\mu n\right) - 2\sin^2\theta_W(\bar{p}\gamma_\mu p)\right]\\
&+ \left(-\tfrac{1}{4} + \sin^2\theta_W\right)4\bar{e}\gamma_\mu e\, g_A\left(\tfrac{1}{2}\bar{p}\gamma_\mu\gamma_5 p - \tfrac{1}{2}\bar{n}\gamma_\mu\gamma_5 n\right)\Big\}
\end{aligned} \tag{12.16}$$

and the coefficients are

$$\begin{aligned}
C_{1p} &= \tfrac{1}{2}(1 - 4\sin^2\theta_W), & C_{1n} &= -\tfrac{1}{2},\\
C_{2p} &= 2\left(-\tfrac{1}{4} + \sin^2\theta_W\right)g_A, & C_{2n} &= -2\left(-\tfrac{1}{4} + \sin^2\theta_W\right)g_A.
\end{aligned} \tag{12.17}$$

**(d)** In the non-relativistic limit, only the time component of the vector current is non-vanishing. It counts the proton and neutron numbers in the nucleus, and we have

$$\begin{aligned}
&\sum_i \langle A, Z|p_i^\dagger p_i|A, Z\rangle = Z\\
&\sum_i \langle A, Z|n_i^\dagger n_i|A, Z\rangle = A - Z = N
\end{aligned} \tag{12.18}$$

and for the combination that appears in the weak neutral current

$$\begin{aligned}
&\langle A, Z|\left[\tfrac{1}{2}\left(1 - 4\sin^2\theta_W\right)\sum_i p_i^\dagger p_i - \tfrac{1}{2}\sum_i n_i^\dagger n_i\right]|A, Z\rangle\\
&\quad = \tfrac{1}{2}\left[\left(1 - 4\sin^2\theta_W\right)Z - N\right] = \tfrac{1}{2}Q_W.
\end{aligned} \tag{12.19}$$

Thus we get

$$\mathcal{L}'_N = \frac{G_F}{2\sqrt{2}}Q_W\, e^\dagger\gamma_5 e.$$

Note that the matrix element of axial vector current $\bar{N}\gamma_\mu\gamma_5 N$ in the non-relativistic limit is proportional to the nuclear spin operator and is smaller than $N$ or $Z$.

## 12.2 Polarization asymmetry of $Z \to \bar{f} f$

The polarization asymmetry (or the *left–right asymmetry*) in the decay of the $Z$ boson into a fermion pair $Z \to f \bar{f}$ is given by

$$A_{LR}(f) = \frac{\Gamma(Z \to f_L \bar{f}_R) - \Gamma(Z \to f_R \bar{f}_L)}{\Gamma(Z \to f_L \bar{f}_R) + \Gamma(Z \to f_R \bar{f}_L)} \tag{12.20}$$

and the neutral current can be written as

$$J_\mu^Z = \sum_f \left[ g_L(f) \left( \bar{f}_L \gamma_\mu f_L \right) + g_R(f) \left( \bar{f}_R \gamma_\mu f_R \right) \right]. \tag{12.21}$$

In this problem we wish to express the asymmetry parameter in terms of the neutral current parameters $g_{L,R}(f)$.

**(a)** Show that the asymmetry parameter $A_{LR}$ can be written as

$$A_{LR}(f) = \frac{(g_L(f))^2 - (g_R(f))^2}{(g_L(f))^2 + (g_R(f))^2}. \tag{12.22}$$

**(b)** Calculate the asymmetry parameter $A_{LR}$ for the decays:

**(i)** $Z \to e\bar{e}$,

**(ii)** $Z \to b\bar{b}$,

**(iii)** $Z \to c\bar{c}$.

For numerical calculation, use $\sin^2 \theta_W = 0.22$.

**Solution to Problem 12.2**

**(a)** In the calculation of the two decay rates in $A_{LR}$, the amplitudes are the same except for the overall couplings ($g_L$ or $g_R$) and $(1 - \gamma_5)$ or $(1 + \gamma_5)$ projections. As there are no $V$–$A$ interference terms in the rates, we have

$$\frac{\Gamma(Z \to f_L \bar{f}_R)}{\Gamma(Z \to f_R \bar{f}_L)} = \frac{(g_L(f))^2}{(g_R(f))^2} \tag{12.23}$$

and thus

$$A_{LR}(f) = \frac{(g_L(f))^2 - (g_R(f))^2}{(g_L(f))^2 + (g_R(f))^2}. \tag{12.24}$$

**(b) (i)** $Z \to e\bar{e}$

$$g_L(e) = -\tfrac{1}{2} + \sin^2 \theta_W = -0.28, \quad g_R(e) = \sin^2 \theta_W = 0.22,$$
$$A_{LR}(e) = 0.2366. \tag{12.25}$$

**(ii)** $Z \to b\bar{b}$

$$g_L(b) = -\tfrac{1}{2} + \tfrac{1}{3} \sin^2 \theta_W = -0.43, \quad g_R(b) = \tfrac{1}{3} \sin^2 \theta_W = 0.071,$$
$$A_{LR}(b) = 0.944. \tag{12.26}$$

**(iii)** $Z \to c\bar{c}$

$$g_L(c) = \tfrac{1}{2} - \tfrac{2}{3} \sin^2 \theta_W = -0.353, \quad g_R(b) = -\tfrac{2}{3} \sin^2 \theta_W = -0.147,$$
$$A_{LR}(c) = 0.7. \tag{12.27}$$

## 12.3 Simple $\tau$-lepton decays

**(a)** Show that to lowest order in QCD and the approximation that all the fermion masses in the final state are negligible we have the following $\tau$-decay branching ratios:

$$B(\tau \to e\nu\bar{\nu}) = B(\tau \to \mu\nu\bar{\nu}) \simeq \frac{1}{5}.$$

**(b)** Calculate the decay rate for $\tau \to \pi\nu$ in terms of the pion decay constant $f_\pi$.

### Solution to Problem 12.3

**(a)** The total decay rate of the tau lepton ($\tau$) is,

$$\Gamma(\tau) = \Gamma(\tau \to e\nu\bar{\nu}) + \Gamma(\tau \to \mu\nu\bar{\nu}) + \Gamma(\tau \to \nu + hadrons). \tag{12.28}$$

From $\mu$–$e$ universality, we get, with the approximation of neglecting final state fermion masses,

$$\Gamma(\tau \to e\nu\bar{\nu}) = \Gamma(\tau \to \mu\nu\bar{\nu}). \tag{12.29}$$

To lowest order in QCD, we get

$$\Gamma(\tau \to \nu + hadrons) = \Gamma(\tau \to \nu + d\bar{u}) + \Gamma(\tau \to \nu + s\bar{u}) \tag{12.30}$$

and

$$\Gamma(\tau \to \nu + d\bar{u}) = |V_{ud}|^2 3\Gamma(\tau \to \mu\nu\bar{\nu}).$$

Thus

$$\Gamma(\tau) = [2 + 3(|V_{ud}|^2 + |V_{us}|^2)]\Gamma(\tau \to e\nu\bar{\nu}). \tag{12.31}$$

From the experimental fact that

$$|V_{ud}|^2 + |V_{us}|^2 \simeq 1 \tag{12.32}$$

we get

$$B(\tau \to \mu\nu\bar{\nu}) = \frac{\Gamma(\tau \to \mu\nu\bar{\nu})}{\Gamma(\tau)} \simeq \frac{1}{5} = B(\tau \to e\nu\bar{\nu}). \tag{12.33}$$

**(b)** The effective Lagrangian for the decay $\tau \to \pi\nu$ is of the form

$$\mathcal{L}_w = \frac{G_F}{\sqrt{2}} V_{ud} \left[\bar{d}\gamma^\mu(1-\gamma_5)u\right]\left[\bar{\nu}_\tau\gamma_\mu(1-\gamma_5)\tau\right]. \tag{12.34}$$

The amplitude is then given by

$$\mathcal{M} = G_F V_{ud} f_\pi q^\mu \bar{\nu}_\tau(k)\gamma_\mu(1-\gamma_5)\tau(p) = G_F V_{ud} f_\pi m_\tau \bar{\nu}_\tau(k)(1-\gamma_5)\tau(p) \tag{12.35}$$

where we have used

$$\langle\pi(q)|\bar{d}\gamma^\mu\gamma_5 u|0\rangle = i\sqrt{2}q_\mu f_\pi \qquad \text{and} \qquad q = p - k. \tag{12.36}$$

The decay rate is given by

$$\Gamma = \frac{1}{2m_\tau}\left(\frac{1}{2}\sum_{spin}|\mathcal{M}|^2\right)(2\pi)^4\delta^4(p-k-q)\frac{d^3k}{(2\pi)^3 2k_0}\frac{d^3q}{(2\pi)^3 2q_0}. \tag{12.37}$$

The phase space is the same as that calculated in Problem 11.3, with appropriate substitutions

$$\begin{aligned}\rho &= \int (2\pi)^4\delta^4(p-k-q)\frac{d^3k}{(2\pi)^3 2k_0}\frac{d^3q}{(2\pi)^3 2q_0}\\ &= \frac{1}{4\pi}\left(\frac{m_\tau^2-m_\pi^2}{2m_\tau^2}\right).\end{aligned} \tag{12.38}$$

The spin average of the matrix element is given by

$$\begin{aligned}\tfrac{1}{2}\sum_{spin}|\mathcal{M}|^2 &= \tfrac{1}{2}\,|G_F V_{ud} f_\pi m_\tau|^2\, Tr[\not{k}\,(1+\gamma_5)\,(\not{p}+m_\tau)\,(1-\gamma_5)]\\ &= 2G_F^2 f_\pi^2 |V_{ud}|^2 m_\tau^2 (2p\cdot k) = 2G_F^2 f_\pi^2|V_{ud}|^2 m_\tau^2\left(m_\tau^2-m_\pi^2\right).\end{aligned}$$

The decay rate is then

$$\begin{aligned}\Gamma &= \frac{1}{m_\tau}G_F^2 f_\pi^2|V_{ud}|^2 m_\tau^2\left(m_\tau^2-m_\pi^2\right)\frac{1}{4\pi}\left(\frac{m_\tau^2-m_\pi^2}{2m_\tau^2}\right)\\ &= \frac{G_F^2}{8\pi}f_\pi^2|V_{ud}|^2 m_\tau^3\left(1-\frac{m_\pi^2}{m_\tau^2}\right)^2.\end{aligned} \tag{12.39}$$

**Remark.** If we compare this with

$$\Gamma(\tau\to\mu\nu\bar{\nu}) = \frac{G_F^2}{192\pi^3}m_\tau^5 \tag{12.40}$$

we get

$$\frac{\Gamma(\tau\to\pi\nu)}{\Gamma\,(\tau\to\mu\nu\bar{\nu})} = \frac{f_\pi^2|V_{ud}|^2(1-m_\pi^2/m_\tau^2)^2(24\pi^2)}{m_\tau^2}\simeq 0.6 \tag{12.41}$$

where we have used $|V_{ud}|\simeq 0.975$ and $f_\pi\simeq 90\,\text{MeV}$. Experimentally, we have

$$\frac{\Gamma(\tau\to\pi\nu)}{\Gamma(\tau\to\mu\nu\bar{\nu})}\simeq 0.66. \tag{12.42}$$

## 12.4 Electron neutrino scatterings

**(a)** Show that the threshold energy for the reaction $\nu_\mu + e^- \to \nu_e + \mu^-$ is $E_\nu = 11\,\text{GeV}$ in the laboratory frame.

**(b)** Show that in the $\nu_e + e^- \to \nu_e + e^-$ elastic scattering, the angle of scattering $\theta_e$ of the electron with respect to the neutrino beam direction satisfies

$$\sin^2\theta_e = \frac{2m_e}{(T_e+2m_e)}\left[1-\frac{T_e}{E_\nu}-\frac{m_eT_e}{2E_\nu^2}\right] \tag{12.43}$$

where $T_e$ is the kinetic energy of the (final) electron.

**Solution to Problem 12.4**

**(a)** Denote the momenta as $\nu_\mu(k_1) + e^-(p_1) \to \nu_e(k_2) + \mu^-(p_2)$, then

$$s = (k_1 + p_1)^2 = m_e^2 + 2k_1 \cdot p_1 = m_e^2 + 2m_e E_\nu \tag{12.44}$$

in the laboratory frame. In order to produce $\mu^-$, we require $s > m_\mu^2$, or

$$E_\nu > \frac{1}{2m_e}\left(m_\mu^2 - m_e^2\right) \simeq 11\,\text{GeV}. \tag{12.45}$$

This calculation shows that in the laboratory frame it takes lots of energy to produce a muon by scattering a neutrino off an (extremely light) electron target.

**(b)** From the momenta assignment, we have in the laboratory frame,

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{p}_2 \tag{12.46}$$

$$k_1 + m_e = k_2 + E_2 \tag{12.47}$$

From eqn (**??**) we have (setting $m_e = m$)

$$\mathbf{k}_2^2 = (\mathbf{k}_1 - \mathbf{p}_2)^2 = \mathbf{k}_1^2 + \mathbf{p}_2^2 - 2k_1 p_2 \cos\theta_e = k_1^2 + E_2^2 - m^2 - 2k_1 p_2 \cos\theta_e. \tag{12.48}$$

From eqn (**??**),

$$k_2^2 = (k_1 + m_e - E_2)^2 = k_1^2 + m^2 + E_2^2 + 2k_1 m - 2mE_2 - 2k_1 E_2. \tag{12.49}$$

Combine these two equations, we get

$$-k_1 p_2 \cos\theta_e = m^2 + k_1 m - (m + k_1)E_2 = (m + k_1)(m - E_2). \tag{12.50}$$

In order to express $p_2$ in terms of $E_2$, we square both sides of this equation:

$$k_1^2\left(E_2^2 - m^2\right)\left(1 - \sin^2\theta_e\right) = (m + k_1)^2(m - E_2)^2 \tag{12.51}$$

or

$$k_1^2(E_2 + m) - (E_2 - m)(m + k_1)^2 = k_1^2(E_2 + m)(\sin^2\theta_e). \tag{12.52}$$

The kinetic energy of the final electron being $T_e = E_2 - m$, the scattering angle satisfies

$$\sin^2\theta_e = \frac{2m_e}{(T_e + 2m_e)}\left[1 - \frac{T_e}{E_\nu} - \frac{m_e T_e}{2E_\nu^2}\right]. \tag{12.53}$$

**Remark.** For the usual neutrino beams we have $E_\nu \gg m_e$ and $T_e \gg m_e$: this formula implies that $\sin^2\theta_e$ is small and

$$\theta_e^2 \simeq \frac{2m_e}{T_e}. \tag{12.54}$$

Thus the electron moves very much in the forward direction and provides a good signature for $\nu e$ quasi-elastic scattering.

## 12.5 CP properties of kaon non-leptonic decays

Consider the $K^0 \to 2\pi, 3\pi$ decays.

**(a)** Show that $|\pi^+\pi^-\rangle$ and $|\pi^0\pi^0\rangle$ are CP even eigenstates.

**(b)** Show that $|\pi^0\pi^0\pi^0\rangle$ is CP odd while the CP eigenvalues of the state $|\pi^+\pi^-\pi^0\rangle$ depend on the orbital angular momentum $l$ of $\pi^0$ with respect to the center of mass of the $\pi^+\pi^-$ system,

$$\mathcal{CP}|\pi^+\pi^-\pi^0\rangle = (-1)^{l+1}|\pi^+\pi^-\pi^0\rangle. \tag{12.55}$$

**Solution to Problem 12.5**

**(a)** Denote the wave function of $\pi^+\pi^-$ by

$$|\pi^+\pi^-\rangle = \psi(\mathbf{r}_1, \mathbf{r}_2) \tag{12.56}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of $\pi^+$ and $\pi^-$, respectively. We can also use the centre of mass and relative coordinates,

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{12.57}$$

to write the wave function as

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \chi(\mathbf{R})\phi(\mathbf{r}) \tag{12.58}$$

where $\chi(\mathbf{R})$ is just a plane wave describing the motion of the centre of mass and is of no interest to us. Under the charge conjugation, we have $\pi^+ \longleftrightarrow \pi^-$, which corresponds to $\boldsymbol{r}_1 \longleftrightarrow \mathbf{r}_2$ or $\mathbf{r} \to -\mathbf{r}$. The effect on the wave function is then

$$\phi(\mathbf{r}) \xrightarrow{C} \phi(-\mathbf{r}) = (-1)^l\phi(\mathbf{r}) \tag{12.59}$$

where $l$ is the orbital angular momentum of the $\pi^+\pi^-$ system. Thus

$$\mathcal{C}|\pi^+\pi^-\rangle = (-1)^l|\pi^+\pi^-\rangle. \tag{12.60}$$

Under the parity, we have $(-1)$ from each of the pion and $\mathbf{r} \to -\mathbf{r}$. Thus we get

$$\mathcal{P}|\pi^+\pi^-\rangle = (-1)^l|\pi^+\pi^-\rangle. \tag{12.61}$$

Combining these two relations we get

$$\mathcal{CP}|\pi^+\pi^-\rangle = (-1)^{2l}|\pi^+\pi^-\rangle = |\pi^+\pi^-\rangle. \tag{12.62}$$

For the $|\pi^0\pi^0\rangle$ state we have, as before,

$$\mathcal{P}|\pi^0\pi^0\rangle = (-1)^l|\pi^0\pi^0\rangle. \tag{12.63}$$

But under the charge conjugation,

$$\mathcal{C}|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle \tag{12.64}$$

because $\pi^0$ is a $C$-even eigenstate. On the other hand, the $\pi^0\pi^0$ system consists of identical bosons and from Bose statistics we should have symmetric wave function under $\mathbf{r}_1 \longleftrightarrow \mathbf{r}_2$ which corresponds to $\mathbf{r} \to -\mathbf{r}$. This means that we can only have $l =$ even states. Thus we also get

$$\mathcal{CP}|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle. \tag{12.65}$$

**(b)** For the $\pi^+\pi^-\pi^0$ state we can write the total angular momentum as

$$\mathbf{J} = \mathbf{J}_{12} + \mathbf{J}_3 \tag{12.66}$$

where $\mathbf{J}_{12}$ is the orbital angular momentum of the $\pi^+\pi^-$ pair and $\mathbf{J}_3$ is the orbital angular momentum of $\pi^0$ with respect to the centre of mass of the $\pi^+\pi^-$ pair. Since $K^0$ has spin-0, we have $\mathbf{J} = 0$, which implies

$$|\mathbf{J}_{12}| = |\mathbf{J}_3|.$$

As we have discussed in Part (a), the $|\pi^+\pi^-\rangle$ state is CP even, irrespective of $l = |J_{12}|$. On the other hand

$$\mathcal{CP}|\pi^0\rangle = (-)(-)^{J_3}|\pi^0\rangle = (-)^{l+1}|\pi^0\rangle \tag{12.67}$$

where $(-)$ comes from the intrinsic parity of $|\pi^0\rangle$ and $(-)^{J_3}$ from the fact that under the parity $\mathbf{r}_3 \to -\mathbf{r}_3$, with $J_3 = |J_{12}| = l$. Then the result is

$$\mathcal{CP}|\pi^+\pi^-\pi^0\rangle = (-1)^{l+1}|\pi^+\pi^-\pi^0\rangle \tag{12.68}$$

For the $3\pi^0$ state the only difference is that by Bose statistics $2\pi^0$ has to be in the $l =$ even state. Then we get

$$\mathcal{CP}|\pi^0\pi^0\pi^0\rangle = -|\pi^0\pi^0\pi^0\rangle. \tag{12.69}$$

**Remark 1.** From the fact that both $K_L \to \pi^+\pi^-$ and $K_L \to \pi^0\pi^0\pi^0$ are seen experimentally, CP symmetry is broken. The fact that the rate for $K_L \to \pi^+\pi^-$ is much smaller than $K_S \to \pi^+\pi^-$ implies that $K_L$ is mostly CP odd state and $K_S$ is mostly CP even state.

**Remark 2.** $K_S \to \pi^0\pi^0\pi^0$ decay also violates the CP conservation if $K_S$ is a pure CP even state. On the other hand, $\pi^+\pi^-\pi^0$ can have both CP even and CP odd wave functions.

## 12.6 $Z \to HH$ is forbidden

Show that in the standard model, the decay of the $Z$ particle into two Higgs bosons, $Z \to HH$, is forbidden by the angular momentum conservation and Bose statistics.

### Solution to Problem 12.6

Because of Bose statistics, the two final-state Higgs particles should be in the spatially symmetric state. This means that the relative angular momentum has to be even, $l = 0, 2, 4\cdots$. But the initial state $Z$ has spin $J = 1$, which cannot go into an even angular momentum state.

**Remark 1.** The same argument applies to $Z \to PP$, where $P$ is a pseudoscalar boson.

**Remark 2.** There are no symmetry argument to forbid the decay $Z \to HHH$ or $Z \to PPP$.

## 12.7 $\Delta I = \frac{1}{2}$ enhancement by short-distance QCD

The effective $\Delta S = 1$ weak Lagrangian is of the form

$$\mathcal{L}_{\Delta S=1} = \frac{4G_F}{\sqrt{2}}(V_{ud}V_{su}O_1 + h.c.) \tag{12.70}$$

where

$$\begin{aligned} O_1 &= (\bar{u}_L\gamma^\mu s_L)\,(\bar{d}_L\gamma_\mu u_L) \\ &= \tfrac{1}{4}[\bar{u}\gamma^\mu(1-\gamma_5)s][\bar{d}\gamma_\mu(1-\gamma_5)u] \end{aligned} \tag{12.71}$$

Show that in the renormalization of the composite operator $O_1$ there is operator mixing between $O_1$ and another operator $O_2$ of the form

$$O_2 = (\bar{u}_L\gamma^\mu u_L)\,(\bar{d}_L\gamma_\mu s_L). \tag{12.72}$$

Also, compute the anomalous dimension matrix for the $O_1$–$O_2$ system. The result should indicate a QCD enhancement of the $\Delta I = \frac{1}{2}$ operator.

### Solution to Problem 12.7

The one-loop QCD corrections to $O_1$ are shown in Fig. 12.1.

For diagrams (a) and (b) in Fig. 12.1, these graphs are just QCD corrections to current, e.g. $(\bar{u}_L\gamma_\mu s_L)$ which has zero anomalous dimension, $\gamma = 0$ because of its partial conservation. This means that these contributions will be cancelled by wave function renormalization which are not shown in Fig. 12.1.

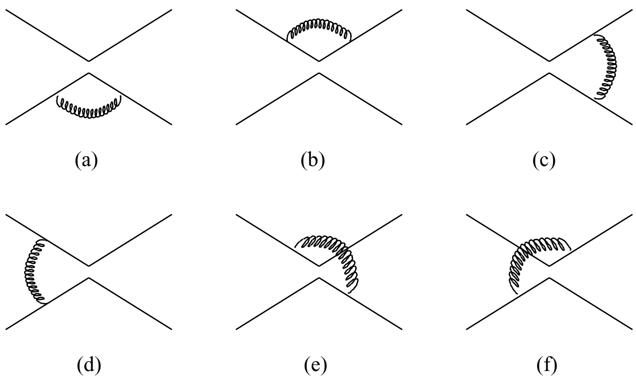


FIG. 12.1. QCD correction to the $(\bar{u}_L\gamma^\mu s_L)(\bar{d}_L\gamma_\mu u_L)$ operator.

For graph (c), we can calculate its contribution by dimensional regularization, where the gauge coupling is of the form $g/(\mu)^{2-d/2}$:

$$\begin{aligned}\mathcal{M}_c &= \int \frac{d^d k}{(2\pi)^4}\left[\bar{u}_L\left(\frac{ig}{\mu^{2-d/2}}\right)\gamma_\alpha t^a \frac{(-i\not{k})}{k^2}\gamma_\mu s_L\right] \\ &\quad \times \frac{(-i)}{k^2}\left[\bar{d}_L\left(\frac{ig}{\mu^{2-d/2}}\right)\gamma^\alpha t^a \frac{(i\not{k})}{k^2}\gamma^\mu u_L\right] \\ &= \frac{ig^2}{(\mu^2)^{2-d/2}}\int \frac{d^d k}{(2\pi)^4}\left(\frac{k^2}{d}\right)\left[\bar{u}_L t^a \gamma_\alpha\gamma_\beta\gamma_\mu s_L\right]\left[\bar{d}_L t^a \gamma^\alpha\gamma^\beta\gamma^\mu u_L\right]\frac{1}{k^6}\end{aligned} \tag{12.73}$$

where we have used the replacement $k_\alpha k_\beta = \frac{k^2}{d} g_{\alpha\beta}$. Using the identity displayed in CL-eqn (A.17),

$$\gamma_\alpha\gamma_\beta\gamma_\mu = g_{\alpha\beta}\gamma_\mu + g_{\beta\mu}\gamma_\alpha - g_{\alpha\mu}\gamma_\beta + i\varepsilon_{\alpha\beta\mu\nu}\gamma^\nu\gamma_5, \tag{12.74}$$

we can reduce the Dirac matrices:

$$[\gamma_\alpha\gamma_\beta\gamma_\mu(1-\gamma_5)]_{\rho\delta}[\gamma^\alpha\gamma^\beta\gamma^\mu(1-\gamma_5)]_{\omega\epsilon} = 16[\gamma_\mu(1-\gamma_5)]_{\rho\delta}[\gamma^\mu(1-\gamma_5)]_{\omega\epsilon}. \tag{12.75}$$

For simplicity we have ignored the complication of defining $\gamma_5$ in the general $d$-dimensions. For the SU(3) colour matrices, we use the identity in CL-eqn (4.134):

$$(t^a)_{ij}(t^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right). \tag{12.76}$$

The amplitude $\mathcal{M}_c$ can then be written as

$$\mathcal{M}_c = \frac{ig^2}{(\mu^2)^{2-d/2}}\left[\frac{1}{d}\int \frac{d^d k}{(2\pi)^4}\frac{1}{k^4}\right] 8\left[(\bar{u}_L\gamma_\mu u_L)(\bar{d}_L\gamma_\mu s_L) - \frac{1}{3}(\bar{u}_L\gamma_\mu s_L)\left(\bar{d}_L\gamma_\mu u_L\right)\right]. \tag{12.77}$$

Performing the integration over $k$,

$$\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^4} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}, \tag{12.78}$$

we get

$$\mathcal{M}_c = \frac{-g^2}{(\mu^2)^{2-d/2}}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{d}(+8)\left[(\bar{u}_L\gamma_\mu u_L)(\bar{d}_L\gamma_\mu s_L) - \frac{1}{3}(\bar{u}_L\gamma_\mu s_L)(\bar{d}_L\gamma_\mu u_L)\right]. \tag{12.79}$$

This shows that the operator $O_2$ mixes with $O_1$ under the renormalization. It is clear that $\mathcal{M}_d$ gives the same contribution as $\mathcal{M}_c$.

For the graph (e) we have

$$\mathcal{M}_e = \frac{ig^2}{(\mu^2)^{2-d/2}} \int \frac{d^d k}{(2\pi)^4} \left(\frac{k^2}{d}\right) \left[\bar{u}_L t^a \gamma_\alpha \gamma_\beta \gamma_\mu s_L\right] \left[\bar{d}_L t^a \gamma^\mu \gamma^\alpha \gamma^\beta u_L\right] \frac{1}{k^6}. \tag{12.80}$$

The ordering of $\gamma$ matrices being different from that of $\mathcal{M}_c$, we get

$$[\gamma_\alpha \gamma_\beta \gamma_\mu (1-\gamma_5)]_{\rho\delta} [\gamma^\mu \gamma^\alpha \gamma^\beta (1-\gamma_5)]_{\omega\epsilon} = -4[\gamma_\mu (1-\gamma_5)]_{\rho\delta} [\gamma^\mu (1-\gamma_5)]_{\omega\epsilon} \tag{12.81}$$

and

$$\mathcal{M}_e = \frac{-g^2}{(\mu^2)^{2-d/2}} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{d} (-2) \left[O_2 - \frac{1}{3} O_1\right]. \tag{12.82}$$

For graph (f), giving the same contribution as (e), the total contribution is

$$\mathcal{M} = 2\mathcal{M}_c + 2\mathcal{M}_e = \frac{-12g^2}{(\mu^2)^{2-d/2}} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\, d} \left[O_2 - \frac{1}{3} O_1\right]. \tag{12.83}$$

For $d \to 4$, we have

$$\frac{\Gamma(2-d/2)}{(4\pi)^{d/2} (\mu^2)^{2-d/2}} \to \frac{1}{(4\pi)^2} \left[\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - \ln \mu^2\right] \tag{12.84}$$

with $\varepsilon = 4 - d$. The counterterm for $O_1$ is then

$$\delta O_1 = \frac{3g^2}{(4\pi)^2} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi)\right) \left(O_2 - \frac{1}{3} O_1\right). \tag{12.85}$$

If we write

$$O_1 + \delta O_1 = Z_{11} O_1 + Z_{12} O_2 \tag{12.86}$$

we get

$$\begin{aligned} Z_{11} &= 1 - \frac{g^2}{(4\pi)^2} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi)\right), \\ Z_{12} &= \frac{3g^2}{(4\pi)^2} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi)\right). \end{aligned} \tag{12.87}$$

It is straightforward to carry out the renormalization for the operator $O_2$ and the result is

$$O_2 + \delta O_2 = Z_{21} O_1 + Z_{22} O_2 \tag{12.88}$$

with

$$Z_{12} = Z_{21}, \qquad Z_{11} = Z_{22}. \tag{12.89}$$

To compute the anomalous dimension, we can use the correspondence between dimensional regulation and the invariant cutoff given in CL-Table 2.1 to write

$$Z_{11} = Z_{22} = 1 + \frac{g^2}{(4\pi)^2}\left(\ln\frac{\Lambda^2}{\mu^2} + \cdots\right),$$
$$Z_{12} = Z_{21} = \frac{-3g^2}{(4\pi)^2}\left(\ln\frac{\Lambda^2}{\mu^2} + \cdots\right). \tag{12.90}$$

Then from the formula for the anomalous dimension matrix

$$\gamma_{ij} = -\frac{1}{2}\frac{\partial}{\partial \ln\Lambda}\ln Z_{ij} \tag{12.91}$$

we get

$$\gamma = \frac{g^2}{(4\pi)^2}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}. \tag{12.92}$$

The eigenvalues are

$$\gamma_+ = \frac{g^2}{(4\pi)^2} \times 2 \qquad \gamma_- = \frac{g^2}{(4\pi)^2} \times (-4), \tag{12.93}$$

with the corresponding eigen operators being

$$O_+ = \frac{1}{2}(O_1 + O_2), \qquad O_- = \frac{1}{2}(O_1 - O_2). \tag{12.94}$$

Or in terms of quarks fields,

$$O_+ = \frac{1}{2}(O_1 + O_2) = \frac{1}{2}[(\bar{u}_L\gamma_\mu s_L)(\bar{d}_L\gamma_\mu u_L) + (\bar{u}_L\gamma_\mu u_L)(\bar{d}_L\gamma_\mu s_L)],$$
$$O_- = \frac{1}{2}(O_1 - O_2) = \frac{1}{2}[(\bar{u}_L\gamma_\mu s_L)(\bar{d}_L\gamma_\mu u_L) - (\bar{u}_L\gamma_\mu u_L)(\bar{d}_L\gamma_\mu s_L)].$$

**Remark.** The operator $O_-$ which is antisymmetric in $\bar{u}_L \leftrightarrow \bar{d}_L$ is a pure $\Delta I = \frac{1}{2}$ operator because $I = 0$ state is antisymmetric in $\bar{u}\bar{d}$. On the other hand, $O_+$ has both $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ operators. Thus QCD corrections enhance the $O_-$ operator ($\gamma_- < 0$), relative to the $O_+$ operator. [Scale factors are raised to the negative powers of $\gamma$, see, for example, CL-eqn (10.148).] But this enhancement of the $\Delta I = \frac{1}{2}$ operator does not seem to be numerically large enough for the explanation of the experimentally observed $\Delta I = \frac{1}{2}$ rule.

## 12.8 Scalar interactions and the equivalence theorem

The standard model Lagrangian for the scalar field is given by

$$\mathcal{L}_s = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{12.95}$$

where

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{12.96}$$

**(a)** Show that if we parametrize the four independent components of the complex doublet field as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \tag{12.97}$$

$\mathcal{L}_s$ is invariant under $O(4)$ rotations, i.e.

$$\phi_i \to \phi_i' = \hat{r}_{ij}\phi_j, \qquad \text{with} \quad i = 1, \ldots, 4 \tag{12.98}$$

where $\hat{r}$ is the four-dimensional rotation matrix $\hat{r}^T\hat{r} = \hat{r}\hat{r}^T = \mathbf{1}$.

**(b)** Show that if we write

$$\boldsymbol{\pi} = (\phi_1, \phi_2, \phi_4), \qquad \sigma = \phi_3, \tag{12.99}$$

then $\mathcal{L}_s$ is the same as the SU(2) × SU (2) sigma-model (without the nucleon).

**(c)** For spontaneous symmetry breaking, we have

$$\phi_3 = \sigma = v + H \qquad \text{with} \quad v^2 = \frac{\mu^2}{\lambda}. \tag{12.100}$$

Write the Lagrangian in terms of $H$ and $\boldsymbol{\pi}$ and find the $H\pi^+\pi^-$, and $Hzz$ couplings, where $\pi^+ = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$ and $z = \pi_3$.

**(d)** Calculate the scattering amplitudes: $\pi^+\pi^- \to zz$, $\pi^+\pi^- \to \pi^+\pi^-$, and $zz \to zz$.

### Solution to Problem 12.8

**(a)** From the parametrization

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \tag{12.101}$$

we get

$$\begin{aligned} \phi^\dagger\phi &= \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = \frac{1}{2}(\boldsymbol{\phi}\cdot\boldsymbol{\phi}) \\ \partial_\mu\phi^\dagger\partial^\mu\phi &= \frac{1}{2}(\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}) \end{aligned} \tag{12.102}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a vector in four-dimensional space. Then the Lagrangian is of the form

$$\mathcal{L}_s = (\partial_\mu\boldsymbol{\phi})^2 + \frac{\mu^2}{2}(\boldsymbol{\phi}\cdot\boldsymbol{\phi}) - \frac{\lambda}{4}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 \tag{12.103}$$

which is clearly $O(4)$ invariant because only the invariant scalar products appear in the Lagrangian.

**(b)** If we break the 4-vector $\phi_i$ into a 3-vector and a scalar, $\boldsymbol{\pi} = (\phi_{1,}\phi_{2,}\phi_4)$ and $\sigma = \phi_3$, the 4-scalar $\boldsymbol{\phi}\cdot\boldsymbol{\phi}$ corresponds to the sum of

$$\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \boldsymbol{\pi}^2 + \sigma^2. \tag{12.104}$$

The Lagrangian is then of the form

$$\mathcal{L}_s = \frac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma)^2\right] + \frac{\mu^2}{2}(\boldsymbol{\pi}^2 + \sigma^2) - \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2)^2 \tag{12.105}$$

which is precisely the SU(2) $\times$ SU(2) $\sigma$-model without the nucleon.

**(c)** To study the consequence of spontaneous symmetry breaking, we write $\sigma = v + H$, then

$$\boldsymbol{\pi}^2 + \sigma^2 = \boldsymbol{\pi}^2 + H^2 + 2Hv + v^2 \tag{12.106}$$

$$\begin{aligned}(\boldsymbol{\pi}^2 + \sigma^2)^2 &= (\boldsymbol{\pi}^2 + H^2)^2 + 4Hv(\boldsymbol{\pi}^2 + H^2) \\ &\quad + 2v^2(\boldsymbol{\pi}^2 + H^2) + 4v^2H^2 + 4v^3H + v^4.\end{aligned} \tag{12.107}$$

The scalar potential is then

$$\begin{aligned}V &= -\frac{\mu^2}{2}(\boldsymbol{\pi}^2 + \sigma^2) + \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2)^2 \\ &= \frac{(2\lambda v^2)}{2}H^2 + \lambda H v(\boldsymbol{\pi}^2 + H^2) + \frac{\lambda}{4}(\boldsymbol{\pi}^2 + H^2)^2.\end{aligned} \tag{12.108}$$

The mass of Higgs is given by

$$m_H^2 = 2\lambda v^2. \tag{12.109}$$

Note that $\boldsymbol{\pi}$s are all massless. It is more convenient to write the Lagrangian as

$$\mathcal{L}_s = \frac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu H)^2\right] - \frac{m_H^2}{2}H^2 - \frac{m_H^2}{2v}H(\boldsymbol{\pi}^2 + H^2) - \frac{m_H^2}{8v^2}(\boldsymbol{\pi}^2 + H^2)^2. \tag{12.110}$$

From $\boldsymbol{\pi}^2 = 2\pi^+\pi^- + zz$, we can read off the $H\pi^+\pi^-$ and $Hzz$ couplings as being $im_H^2/v$ and $im_H^2/v$, respectively. The decay rate for $H \to \pi^+\pi^-$ is

$$\begin{aligned}\Gamma(H \to \pi^+\pi^-) &= \left(\frac{m_H^2}{v}\right)^2 \frac{1}{2m_H} \int (2\pi)^4 \delta^4(p - k - k') \frac{d^3k}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2E'} \\ &= \frac{m_H^3}{2v^2}\frac{1}{8\pi} = \frac{m_H^3 G_F}{8\sqrt{2}\pi}\end{aligned} \tag{12.111}$$

where we have used the vacuum expectation value $v^{-2} = \sqrt{2}G_F$. We see that in the limit $m_H \gg M_W$, this agrees with the decay rate $\Gamma(H \to W^+W^-)$ calculated in the next problem.

Similarly, we get for the decay $H \to zz$,

$$\Gamma(H \to zz) = \frac{m_H^3}{2v^2}\frac{1}{8\pi} \times \frac{1}{2} = \frac{m_H^3 G_F}{16\sqrt{2}\pi}. \tag{12.112}$$

**Remark.** This is the essence of the equivalence theorem: one can replace $W_L$ and $Z_L$ with the corresponding 'would-be-Goldstone bosons' in the limit $m_H \gg M_W$ and $M_Z$.

**(d)** Would-be-Goldstone boson scattering amplitudes

**(i)** $\pi^+\pi^- \to zz$

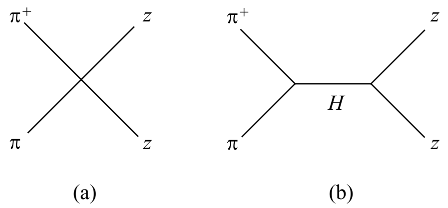



FIG. 12.2. Tree diagrams for $\pi^+\pi^- \to zz$.

The amplitudes for these diagrams are

$$\mathcal{M}^{(a)} = \frac{-im_H^2}{v^2}, \qquad \mathcal{M}^{(b)} = \left(\frac{-im_H^2}{v}\right)^2 \frac{i}{s - m_H^2} \tag{12.113}$$

and the sum is

$$\mathcal{M} = \mathcal{M}^{(a)} + \mathcal{M}^{(b)} = \frac{-im_H^2}{v^2}\left[\frac{s}{s - m_H^2}\right]. \tag{12.114}$$

**Remark.** The amplitude $\mathcal{M}$ vanishes as $s \to 0$, as expected from the usual low energy theorem for the Goldstone boson.

**(ii)** $zz \to zz$

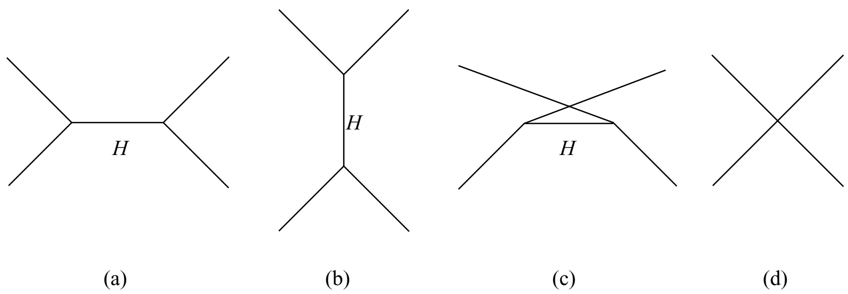



FIG. 12.3. Tree diagrams for $zz \to zz$.

The amplitudes for these diagrams are

$$\mathcal{M}^{(a)} = \left(\frac{-im_H^2}{v}\right)^2 \frac{i}{s - m_H^2}, \qquad \mathcal{M}^{(b)} = \left(\frac{-im_H^2}{v}\right)^2 \frac{i}{t - m_H^2} \tag{12.115}$$

$$\mathcal{M}^{(c)} = \left(\frac{-im_H^2}{v}\right)^2 \frac{i}{u - m_H^2}, \qquad \mathcal{M}^{(d)} = \left(\frac{-im_H^2}{8v^2}\right) 24 \tag{12.116}$$

and the sum is

$$\mathcal{M} = \sum_i \mathcal{M}^{(i)} = -\frac{im_H^2}{v^2}\left[\frac{s}{s-m_H^2} + \frac{t}{t-m_H^2} + \frac{u}{u-m_H^2}\right] \tag{12.117}$$

where $t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$.

**(iii)** $\pi^+\pi^- \to \pi^+\pi^-$

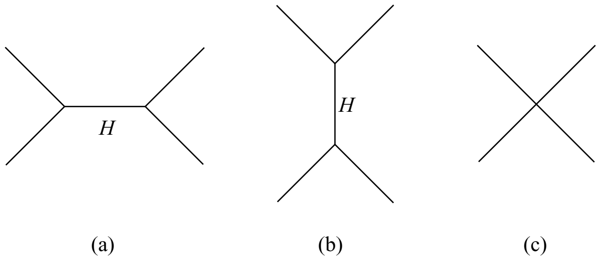



FIG. 12.4. Tree diagrams for $\pi^+\pi^- \to \pi^+\pi^-$.

The amplitudes are

$$\mathcal{M}^{(a)} = \left(\frac{-im_H^2}{v}\right)^2 \frac{i}{s-m_H^2}, \qquad \mathcal{M}^{(b)} = \left(\frac{-im_H^2}{v}\right)^2 \frac{i}{t-m_H^2} \tag{12.118}$$

$$\mathcal{M}^{(c)} = \frac{-im_H^2}{8v^2} 4 \times 2 \times 2 \tag{12.119}$$

and the sum

$$\mathcal{M} = -\frac{im_H^2}{v^2}\left[\frac{s}{s-m_H^2} + \frac{t}{t-m_H^2}\right]. \tag{12.120}$$

**Remark.** These simple results for the scattering of Goldstone bosons can be used, through the equivalent theorem, to get the amplitudes for the longitudinal gauge boson scattering.

## 12.9 Two-body decays of a heavy Higgs boson

Suppose that in the standard model the Higgs particle is heavy so that $m_H > 2M_Z$. Calculate the decay rates for the following modes:

**(a)** $H \to W^+W^-$

**(b)** $H \to ZZ$

**(c)** $H \to t\bar{t}$.

**(d)** Show that in the limit $m_H \gg M_W$, the decay $H \to W^+W^-$ is dominated by $H \to W_L^+ W_L^-$, where $W_L^\pm$ are the longitudinal components of $W^\pm$.

**Solution to Problem 12.9**

**(a)** $H \to W^+W^-$
We can read off the $HWW$ coupling from CL-eqn (12.165) and write the amplitude as

$$\mathcal{M}_a = -igM_W(\varepsilon_1 \cdot \varepsilon_2). \tag{12.121}$$

Then

$$\sum_{spin} |\mathcal{M}_a|^2 = g^2 M_W^2 \sum_{spin} (\varepsilon_1 \cdot \varepsilon_2)^2 = g^2 M_W^2 \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}\right)\left(-g_{\mu\nu} + \frac{k'_\mu k'_\nu}{M_W^2}\right)$$
$$= g^2 M_W^2 \left(2 + \frac{(k \cdot k')^2}{M_W^4}\right). \tag{12.122}$$

From $m_H^2 = (k + k')^2$, we get $k \cdot k' = \frac{1}{2}\left(m_H^2 - 2M_W^2\right)$. Then

$$\sum_{spin} |\mathcal{M}_a|^2 = \frac{g^2}{4M_W^2}\left(m_H^4 - 4m_H^2 M_W^2 + 12M_W^4\right)$$
$$= \frac{2G_F}{\sqrt{2}} m_H^4 \left(1 - \frac{4M_W^2}{m_H^2} + 12\frac{M_W^4}{m_H^4}\right). \tag{12.123}$$

The decay rate is then

$$\Gamma = \frac{1}{2m_H}\int (2\pi)^4 \delta^4(p - k - k') \frac{d^3k}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2E'} \sum_{spin} |\mathcal{M}_a|^2. \tag{12.124}$$

The phase space in the rest frame of $H$ is

$$\rho = \int (2\pi)^4 \delta^4(p - k - k') \frac{d^3k}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2E'}$$
$$= \frac{1}{(2\pi)^2}\int \delta(m_H - E - E') \frac{d^3k}{2E2E'}. \tag{12.125}$$

Write

$$d^3k = k^2 dk 4\pi = 4\pi k E dE. \tag{12.126}$$

We get

$$\rho = \frac{1}{4\pi}\int \delta(m_H - 2E) \frac{kdE}{E} = \frac{k}{4\pi m_H}. \tag{12.127}$$

The momentum $k$ can be calculated as follows:

$$E = \tfrac{1}{2} m_H = (k^2 + M_W^2)^{1/2} \;\;\Rightarrow\;\; k = \tfrac{1}{2}(m_H^2 - 4M_W^2)^{1/2}. \tag{12.128}$$

The phase space is then

$$\rho = \frac{1}{8\pi}\left(1 - \frac{4M_W^2}{m_H^2}\right)^{1/2}. \tag{12.129}$$

Note that the result here for the two-body phase space is different from that calculated in Problem 12.3, because of the equal masses in the final state here. The decay rate is then

$$\Gamma(H \to WW) = \frac{m_H^3 G_F}{8\sqrt{2}\pi}\left(1 - \frac{4M_W^2}{m_H^2}\right)^{1/2}\left(1 - \frac{4M_W^2}{m_H^2} + 12\frac{M_W^4}{m_H^4}\right). \tag{12.130}$$

Note that in the limit of $m_H \gg M_W$, this is the same as eqn (**??**)—as an expression of the equivalence theorem.

**(b)** $H \to ZZ$
The amplitude is given by

$$\mathcal{M}_b = -i\frac{gM_Z}{\cos\theta_W}(\varepsilon_1 \cdot \varepsilon_2). \tag{12.131}$$

The phase space having an extra factor of $\frac{1}{2}$, because of the identical particles in the decay products, the decay rate is then

$$\Gamma(H \to ZZ) = \frac{m_H^3 G_F}{16\sqrt{2}\pi}\left(1 - \frac{4M_Z^2}{m_H^2}\right)^{1/2}\left(1 - \frac{4M_Z^2}{m_H^2} + 12\frac{M_Z^4}{m_H^4}\right). \tag{12.132}$$

This is the same as eqn (**??**), if $m_H \gg M_Z$.

**(c)** $H \to t\bar{t}$
The amplitude is given by

$$\mathcal{M}_c = \frac{-im_t}{v}\bar{v}(k')u(k) \tag{12.133}$$

and

$$\begin{aligned}\sum_{spin}|\mathcal{M}_c|^2 &= \frac{m_t^2}{v^2}Tr[(\not{k} + m_t)(\not{k}' - m_t)] \\ &= \frac{4m_t^2}{v^2}\left(k \cdot k' - m_t^2\right) \\ &= 2\sqrt{2}G_F m_t^2 m_H^2\left(1 - \frac{4m_t^2}{m_H^2}\right)\end{aligned} \tag{12.134}$$

where we have used $k \cdot k' = \frac{1}{2}\left(m_H^2 - 2m_t^2\right)$ and $v^{-2} = \sqrt{2}G_F$. With the phase space

$$\rho = \frac{1}{8\pi}\left(1 - \frac{4m_t^2}{m_H^2}\right)^{1/2} \tag{12.135}$$

we get for the decay rate, which should include a colour factor of 3,

$$\Gamma(H \to t\bar{t}) = \frac{m_H m_t^2 G_F}{4\sqrt{2}\pi}\left(1 - \frac{4m_t^2}{m_H^2}\right)^{3/2} \times 3. \tag{12.136}$$

**Remark.** Since $m_t$ is now measured to be around 180 GeV, the decay $H \to t\bar{t}$ is kinematically allowed if $m_H > 370$ GeV. For the case $m_H^2 \gg m_t^2$, the decay $H \to WW$ and $ZZ$ will dominate over $H \to t\bar{t}$.

**(d)** As we have mentioned in the text (CL-p. 343), the longitudinal polarization is of the form

$$\varepsilon_L^\mu(k) = \frac{1}{M_W}(k, 0, 0, E) \tag{12.137}$$

which, in the high energy limit $k \gg M_W$, takes the form

$$\varepsilon_L^\mu(k) = \frac{k^\mu}{M_W} + O\left(\frac{M_W}{E}\right). \tag{12.138}$$

The matrix element for $H \to W_L^+ W_L^-$ is then

$$\mathcal{M}_{LL} = -ig\varepsilon_L(k) \cdot \varepsilon_L(k') \tag{12.139}$$

and

$$\begin{aligned} |\mathcal{M}_{LL}|^2 &= g^2 M_W^2 |\varepsilon_L(k) \cdot \varepsilon_L(k')|^2 \simeq g^2 M_W^2 \frac{(k \cdot k')^2}{M_W^4} \\ &= \frac{g^2}{M_W^2}\left[\frac{1}{2}\left(m_H^2 - m_t^2\right)\right]^2 \simeq \frac{g^2 m_H^4}{4M_W^2}. \end{aligned} \tag{12.140}$$

This is exactly the same answer for the decay $H \to W^+W^-$ in the limit $m_H \gg M_W$. This shows that $H \to W^+W^-$ is dominated by $H \to W_L^+ W_L^-$.

**Remark 1.** Since the difference between total decay and the decay $H \to W_L^+ W_L^-$ is of the order of $M_W^2/m_H^2$, the decay of the Higgs boson into the transverse modes, $H \to W_T^+ W_T^-$, is suppressed by $O\left(M_W^2/m_H^2\right)$.

**Remark 2.** We can translate this into a relation between effective coupling constants:

$$\frac{g_{HW_LW_L}}{g_{HW_TW_T}} \simeq \left(\frac{m_H}{M_W}\right) \qquad \text{in the limit } m_H \gg M_W. \tag{12.141}$$

This means that in this limit, the Higgs coupling to $W_L$ is much larger than that to $W_T$. On the other hand, $W_L$ comes originally from the scalar fields. Thus the physics of gauge bosons $W_L$ and Higgs field $H$ can be described in terms of scalar self-interaction present in the original Lagrangian. This is the basis of the *Equivalence Theorem* (between longitudinal gauge bosons and the scalar Higgs mode) (see, for example, Cornwall *et al*. 1974).

**Remark 3.** The same argument applies to the decay mode $H \to Z_L Z_L$ which will dominate over $H \to Z_T Z_T$.

# 13 Topics in flavourdynamics

## 13.1 Anomaly-free condition in a technicolour theory

Consider the simplest technicolour theory with one left-handed doublet as given in CL-p. 405. Show that to avoid the anomaly in the $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ gauge group, we need to make the charge assignment of techniquarks:

$$Q(U) = \tfrac{1}{2} \qquad Q(D) = -\tfrac{1}{2}. \tag{13.1}$$

**Solution to Problem 13.1**

Consider the triangle graph in Fig. 13.1 which is the source of the anomaly.

Since there is no anomaly in the SU(2) group, we first consider the case with only one U(1) vertex, in which the graph is proportional to

$$Tr(\tau_i \tau_j Y). \tag{13.2}$$

Because of the presence of $\tau$s only the doublet members can contribute to this $Tr$ sum. $Y$ commutes with $\tau_i$ we can write

$$Tr(\tau_i \tau_j Y) = Tr(\tau_i Y \tau_j) = Tr(\tau_j \tau_i Y) = \tfrac{1}{2} Tr(\{\tau_i, \tau_j\} Y) = \delta_{ij} Tr(Y) \tag{13.3}$$

On the other hand, $Y$ takes the same value for both members of the doublet. This means that the absence of anomaly requires that $Y = 0$ for the doublet. From the relation $Q = T_3 + (Y/2)$, we see that

$$Q(U_L) = \tfrac{1}{2} \qquad Q(D_L) = -\tfrac{1}{2}. \tag{13.4}$$

This charge assignment implies that in the right-handed singlet sector

$$Y(U_R) = \tfrac{1}{2} = -Y(D_R). \tag{13.5}$$

For the case of two U(1) vertices, the coefficient of the triangle graph vanishes because $Tr(\tau_i) = 0$. For the case of three U(1) vertices, only the right-handed

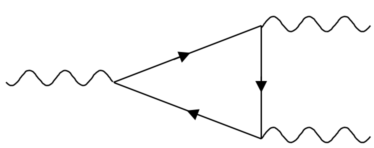


FIG. 13.1. The triangle diagram with a fermion loop.

singlet will contribute and the coefficient is proportional to $Tr(Y^3)$. This vanishes as $U_R$ and $D_R$ contributions cancel because $Y(U_R) = \frac{1}{2} = -Y(D_R)$.

**Remark.** One way to avoid this strange (unfamiliar) charge assignment is to introduce a technilepton as in the standard model

$$\begin{pmatrix} U \\ D \end{pmatrix}_L \begin{pmatrix} N \\ E \end{pmatrix}_L U_R, D_R, N_R, E_R \tag{13.6}$$

with the usual charge assignments:

$$Q(U) = \tfrac{2}{3}, \qquad Q(D) = -\tfrac{1}{3}, \qquad Q(N) = 0, \qquad Q(E) = -1. \tag{13.7}$$

## 13.2 Pseudo-Goldstone bosons in a technicolour model

Consider the case of one generation of technifermions

$$\begin{pmatrix} U \\ D \end{pmatrix}_L \begin{pmatrix} N \\ E \end{pmatrix}_L U_R, D_R, N_R, E_R \tag{13.8}$$

where the charge assignment is the same as the ordinary fermions in the standard model.

**(a)** If one turns off all except technicolour interactions, show that the model has an $SU(8)_L \times SU(8)_R$ global chiral symmetry.

**(b)** Suppose the technifermion condensate breaks this symmetry down to $SU(8)_{L+R}$. Show that there are 60 new Goldstone bosons besides those which were removed by gauge bosons.

**(c)** Show that these Goldstone bosons will become massive if we turn on the gauge interaction.

### Solution to Problem 13.2

**(a)** Since $U$ and $D$ are $SU(3)_C$ triplets, we have eight left-handed and eight right-handed technifermions. Thus the global flavour symmetry is $SU(8)_L \times SU(8)_R$.

**(b)** In the symmetry-breaking $SU(8)_L \times SU(8)_R \rightarrow SU(8)_{L+R}$, we get $8^2 - 1 = 63$ Goldstone bosons, of which three are removed by gauge bosons to break the gauge symmetry from $SU(2) \times U(1)$ down to $U(1)_{em}$.

**(c)** Since the gauge interactions of $SU(2)_L \times U(1)_Y$ do not have the chiral $SU(8)_L \times SU(8)_R$ symmetry, these Goldstone will become massive when the gauge interactions are turned on. These particles are usually referred to as *Pseudo-Goldstone Bosons*. One expects their masses to be of the order of $gM_W$ and they can be as low as a few GeV. If they are as light as a few tens of GeV, they should have been seen by now. This is one of the difficulties in constructing a phenomenologically viable model based on the technicolour idea.

## 13.3 Properties of Majorana fermions

In the standard representation, the solution to the Dirac equation in the momentum space can be chosen to be of the form

$$\begin{aligned} u(p,s) &= (E+m)^{1/2}\begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\end{pmatrix}\chi(s) \\ v(p,s) &= (E+m)^{1/2}\begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \\ 1\end{pmatrix}\chi^c(s) \end{aligned} \tag{13.9}$$

with

$$\chi\left(\tfrac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi\left(-\tfrac{1}{2}\right) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13.10}$$

and

$$\chi^c\left(\tfrac{1}{2}\right) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi^c\left(-\tfrac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{13.11}$$

Note that for convenience we have changed the spin wave function from $\chi(s)$ to $\chi^c(s)$ in the $v$-spinor. Also there is an external minus sign in $\chi\left(-\frac{1}{2}\right)$.

**(a)** Show that with these choices, the charge conjugation will just interchange the $u$- and $v$-spinors, i.e.

$$\begin{aligned} u^c(p,s) &= i\gamma_2 u^*(p,s) = v(p,s) \\ v^c(p,s) &= i\gamma_2 v^*(p,s) = u(p,s). \end{aligned} \tag{13.12}$$

**(b)** If we write the free Dirac field operator as

$$\psi(x) = \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}}\left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right] \tag{13.13}$$

show that the Majorana field $\psi_M$ defined by

$$\psi_M = \frac{1}{\sqrt{2}}(\psi + \psi^c) \tag{13.14}$$

can be decomposed as

$$\psi_M(x) = \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}}\left[b_M(p,s)u(p,s)e^{-ip\cdot x} + b_M^\dagger(p,s)v(p,s)e^{ip\cdot x}\right] \tag{13.15}$$

with

$$b_M(p,s) = \frac{1}{\sqrt{2}}(b(p,s) + d(p,s)). \tag{13.16}$$

Also, compute the anticommutator

$$\{b_M(p,s), b_M^\dagger(p',s')\}. \tag{13.17}$$

**(c)** Show that

$$\bar{u}(p,s)\gamma_\mu u(p',s') = \bar{v}(p',s')\gamma_\mu v(p,s)$$
$$\bar{v}(p,s)\gamma_\mu u(p',s') = \bar{u}(p',s')\gamma_\mu v(p,s) \tag{13.18}$$

so that

$$\bar{\psi}_M\gamma_\mu\psi_M = 0, \qquad \bar{\psi}_M\sigma_{\mu\nu}\psi_M = 0. \tag{13.19}$$

**Solution to Problem 13.3**

**(a)** Write out the conjugate spinor in the standard representation for the $\gamma$-matrices:

$$\begin{aligned}
u^c(p,s) &= i\gamma_2 u^*(p,s) = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} N_E \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma}^*\cdot\mathbf{p}}{E+m} \end{pmatrix} \chi^*(s) \\
&= N_E \begin{pmatrix} i\sigma_2\left(\dfrac{\boldsymbol{\sigma}^*\cdot\mathbf{p}}{E+m}\right) \\ -i\sigma_2 \end{pmatrix} \chi^*(s) = \begin{pmatrix} i\sigma_2\left(\dfrac{\boldsymbol{\sigma}^*\cdot\mathbf{p}}{E+m}\right)(-i\sigma_2) \\ -1 \end{pmatrix} i\sigma_2\chi^*(s) \\
&= -N_E \begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \\ 1 \end{pmatrix} i\sigma_2\chi^*(s) = N_E \begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \\ 1 \end{pmatrix} \chi^c(s) = v(p,s)
\end{aligned} \tag{13.20}$$

where we have used the relations

$$(i\sigma_2)(\boldsymbol{\sigma}^*)(-i\sigma_2) = -\boldsymbol{\sigma}, \qquad -i\sigma_2\chi^*(s) = \chi^c(s). \tag{13.21}$$

**(b)** From the decomposition of a Dirac field

$$\psi(x) = \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}} \left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right] \tag{13.22}$$

we get

$$\begin{aligned}
\psi^c(x) &= i\gamma_2\psi^\dagger(x) \\
&= \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}} \left[b^\dagger(p,s)u^c(p,s)e^{ip\cdot x} + d(p,s)v^c(p,s)e^{-ip\cdot x}\right] \\
&= \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}} \left[d(p,s)u(p,s)e^{-ip\cdot x} + b^\dagger(p,s)v(p,s)e^{ip\cdot x}\right].
\end{aligned}$$

Thus we find that

$$\begin{aligned}
\psi_M &= \frac{1}{\sqrt{2}}[\psi(x) + \psi^c(x)] \\
&= \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}} \left[b_M(p,s)u(p,s)e^{-ip\cdot x} + b_M^\dagger(p,s)v(p,s)e^{ip\cdot x}\right]
\end{aligned} \tag{13.23}$$

with

$$b_M = \frac{1}{\sqrt{2}}[b(p,s) + d(p,s)]. \tag{13.24}$$

The anticommutator can be computed

$$\begin{aligned}\left\{b_M(p,s), b_M^\dagger(p',s')\right\} &= \tfrac{1}{2}\left\{b(p,s) + d(p,s), b^\dagger(p',s') + d^\dagger(p',s')\right\} \\ &= \tfrac{1}{2}\left\{b(p,s), b^\dagger(p',s')\right\} + \tfrac{1}{2}\left\{d(p,s), d^\dagger(p',s')\right\} \\ &= \delta_{ss'}\delta^3(\mathbf{p} - \mathbf{p}'). \end{aligned} \tag{13.25}$$

Thus $b_M$ and $b_M^\dagger$ are just the usual destruction and creation operators.

**(c)** From part (a), we have

$$u^c(p,s) = v(p,s) = i\gamma_2 u^*(p,s) \tag{13.26}$$

or

$$v^\dagger(p,s) = u^T(p,s)(-i\gamma_2^\dagger) = u^T(p,s)(i\gamma_2). \tag{13.27}$$

Then

$$\begin{aligned}\bar{v}(p,s)\gamma_\mu v(p',s') &= v^\dagger(p,s)\gamma_0\gamma_\mu v(p',s') \\ &= u^T(p,s)(i\gamma_2)(\gamma_0\gamma_\mu)(i\gamma_2)u^*(p',s') \\ &= u^\dagger(p',s')(i\gamma_2)^T(\gamma_\mu^T\gamma_0^T)(i\gamma_2)^T u(p,s) \\ &= \bar{u}(p',s')(-i\gamma_2)(\gamma_0\gamma_\mu^T\gamma_0)(i\gamma_2)u(p,s) \\ &= \bar{u}(p',s')(-i\gamma_2)\gamma_\mu^*(i\gamma_2)u(p,s) = \bar{u}(p',s')\gamma_\mu u(p,s) \end{aligned} \tag{13.28}$$

where we have used the relations

$$\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu, \qquad \gamma_0\gamma_\mu^T\gamma_0 = \gamma_\mu^*, \qquad (i\gamma_2)\gamma_\mu^\dagger(-i\gamma_2) = \gamma_\mu. \tag{13.29}$$

Alternatively, we can write

$$v(p,s) = u^c(p,s) = i\gamma_2 u^*(p,s) = i\gamma_2\gamma_0\gamma_0 u^*(p,s) = \mathcal{C}\bar{u}^T \tag{13.30}$$

where

$$\mathcal{C} = i\gamma_2\gamma_0, \qquad \mathcal{C}^{-1} = \mathcal{C}^T = \mathcal{C}^\dagger = -\mathcal{C}, \qquad \mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = -\gamma_\mu. \tag{13.31}$$

Similarly

$$v^\dagger(p,s) = u^T(p,s)i\gamma_2, \qquad \bar{v}(p,s) = u^T(p,s)i\gamma_2\gamma_0 = u^T(p,s)\mathcal{C}. \tag{13.32}$$

Thus

$$\begin{aligned}\bar{v}(p,s)\gamma_\mu v(p',s') &= u^T(p,s)\mathcal{C}\gamma_\mu\mathcal{C}\bar{u}^T(p',s') = u^T(p,s)\gamma_\mu^T\bar{u}^T(p',s') \\ &= \bar{u}(p',s')\gamma_\mu u(p,s). \end{aligned} \tag{13.33}$$

More generally,

$$\bar{v}(p,s)\Gamma v(p',s') = u^T(p,s)\mathcal{C}\Gamma\mathcal{C}\bar{u}^T(p',s') = -u^T(p,s)\mathcal{C}\Gamma\mathcal{C}^{-1}\bar{u}^T(p',s')$$
$$= \bar{u}(p',s')\Gamma^c u(p,s) \tag{13.34}$$

where

$$\Gamma^c = (-\mathcal{C}\Gamma\mathcal{C}^{-1})^T. \tag{13.35}$$

For the various cases,

$$\Gamma = \gamma_\mu \qquad \gamma_\mu^c = \left(-\mathcal{C}\gamma_\mu^{-1}\mathcal{C}\right)^T = \left(\gamma_\mu^T\right)^T = \gamma_\mu$$
$$\Gamma = \gamma_\mu\gamma_5 \quad \left(\gamma_\mu\gamma_5\right)^c = \left(-\mathcal{C}\gamma_\mu\gamma_5^{-1}\mathcal{C}\right)^T = \left(\gamma_5^T\gamma_\mu^T\right)^T = \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$$
$$\Gamma = \gamma_5 \qquad \gamma_5^c = \left(-\mathcal{C}\gamma_5^{-1}\mathcal{C}\right)^T = -\gamma_5$$

$$\Gamma = 1 \qquad 1^c = (-\mathcal{C}\mathcal{C}^{-1})^T = -1$$
$$\Gamma = \sigma_{\mu\nu} \quad \sigma_{\mu\nu}^c = \left(-\mathcal{C}\sigma_{\mu\nu}^{-1}\mathcal{C}\right)^T = -\frac{i}{2}([(\gamma_\mu)^T,(\gamma_\nu)^T])^T = -\frac{i}{2}[\gamma_\nu,\gamma_\mu]$$
$$= \sigma_{\mu\nu}. \tag{13.36}$$

From the Majorana field expansion given in eqn (13.23),

$$\psi_M(x) = \sum_s \int \frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}}\left[b_M(p,s)u(p,s)e^{-ip\cdot x} + b_M^\dagger(p,s)v(p,s)e^{ip\cdot x}\right]. \tag{13.37}$$

$$\bar{\psi}_M\gamma_\mu\psi_M = \sum_{s,s'}\int\frac{d^3p}{\left[(2\pi)^3 2E_p\right]^{1/2}}\int\frac{d^3p'}{\left[(2\pi)^3 2E_p'\right]^{1/2}}$$
$$\times [b_M^\dagger(p,s)b_M(p',s')\bar{u}(p,s)\gamma_\mu u(p',s')e^{-i(p-p')\cdot x}$$
$$+ b_M(p,s)b_M^\dagger(p',s')\bar{v}(p,s)\gamma_\mu v(p',s')e^{i(p-p')\cdot x}$$
$$+ b_M^\dagger(p,s)b_M^\dagger(p',s')\bar{u}(p,s)\gamma_\mu v(p',s')e^{-i(p+p')\cdot x}$$
$$+ b_M(p,s)b_M(p',s')\bar{v}(p,s)\gamma_\mu u(p',s')e^{i(p+p')\cdot x}]. \tag{13.38}$$

We can write the second term as

$$\sum_{s,s'}\int\int b_M(p,s)b_M^\dagger(p',s')\bar{u}(p',s')\gamma_\mu u(p,s)e^{-i(p'-p)\cdot x}$$
$$= \sum_{s,s'}\int\int b_M(p',s')b_M^\dagger(p,s)\bar{u}(p,s)\gamma_\mu u(p',s')e^{-i(p-p')\cdot x}$$

where we have interchanged $(s,\, p) \longleftrightarrow (s',\, p')$. Then the first and second terms combine into

$$\sum_{s,s'} \int \frac{d^3 p}{\left[(2\pi)^3 2E_p\right]^{1/2}} \frac{d^3 p'}{\left[(2\pi)^3 2E'_p\right]^{1/2}}$$
$$\times \{b_M^\dagger(p,s), b_M(p',s')\}\bar{u}(p,s)\gamma_\mu u(p',s')e^{-i(p-p')\cdot x}$$
$$= \sum_s \int \frac{d^3 p}{\left[(2\pi)^3 2E_p\right]^{1/2}} \bar{u}(p,s)\gamma_\mu u(p,s). \tag{13.39}$$

This is a c-number and can be removed by normal ordering. The third term can be written as

$$\sum_{s,s'} \int\int b_M^\dagger(p,s) b_M^\dagger(p',s')\bar{u}(p',s')\gamma_\mu v(p,s)e^{-i(p'+p)\cdot x}$$
$$= \sum_{s,s'} \int\int \tfrac{1}{2}\{b_M^\dagger(p,s) b_M^\dagger(p',s')\}\bar{u}(p',s')\gamma_\mu v(p,s)e^{-i(p'+p)\cdot x} = 0. \tag{13.40}$$

Similarly, the fourth term is also zero. In an entirely analogous way, we can show that $\bar{\psi}_M \sigma_{\mu\nu}\psi_M = 0$.

## 13.4 $\mu \to e\gamma$ and heavy neutrinos

Consider the decay $\mu \to e\gamma$ in the same model as in CL-Section 13.3, but without the assumption that all neutrinos are much lighter than the W boson.

**(a)** Show that the branching ratio is of the form

$$B(\mu \to e\gamma) = \frac{3\alpha}{32\pi}(\delta_\nu^{'})^2 \tag{13.41}$$

where

$$\delta_\nu^{'} = 2\sum_i U_{ei}^* U_{\mu i}\, g\left(\frac{m_i^2}{M_W^2}\right) \tag{13.42}$$

with

$$g(x) = \int_0^1 \frac{(1-\alpha)d\alpha}{(1-\alpha)+\alpha x}[2(1-\alpha)(2-\alpha)+\alpha(1+\alpha)x]. \tag{13.43}$$

**(b)** Show that, for the case $m_i^2 \ll M_W^2$, this reproduces the result in CL-eqn (13.115).

**(c)** Show that $g(0) \neq g(\infty)$ and, if $m_3 \gg M_W$ and $m_{1,2} \ll M_W$, the result is of the form

$$B(\mu \to e\gamma) = \frac{3\alpha}{32\pi}\{2[g(\infty)-g(0)]U_{e\mu}^* U_{\mu 3}\}^2. \tag{13.44}$$

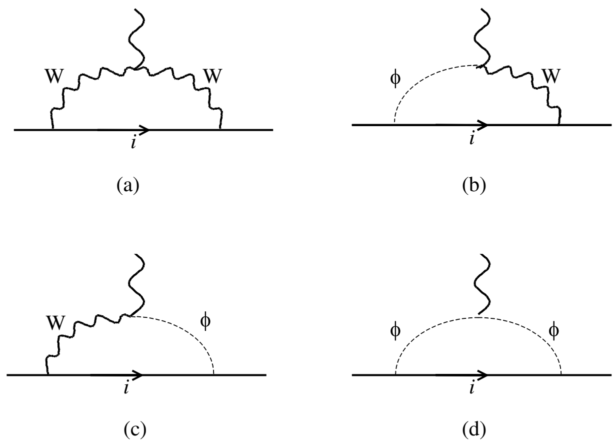



FIG. 13.2. $\mu \to e\gamma$ via neutrino mixings.

**Solution to Problem 13.4**

**(a)** We will carry out the calculation in the 't Hooft–Feynman gauge. First, consider the diagram (a).

$$T(a) = -i \sum_i \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p-q) \left(\frac{ig}{2\sqrt{2}}\right) U_{ei}^* \gamma_\mu (1-\gamma_5) \frac{i}{\gamma \cdot (p+k) - m_i}$$
$$\times \left(\frac{ig}{2\sqrt{2}}\right) U_{\mu i} \gamma_\nu (1-\gamma_5) u_\mu(p) \frac{\left(-ig^{\nu\beta}\right)}{\left(k^2 - M_W^2\right)}$$
$$\times \frac{(-ig^{\mu\alpha})}{[(k+q)^2 - M_W^2]} (-ie\Gamma_{\alpha\beta})$$

where

$$\Gamma_{\alpha\beta} = (2k \cdot \varepsilon) g_{\alpha\beta} - (k+2q)_\beta \varepsilon_\alpha - (k-q)_\alpha \varepsilon_\beta. \tag{13.45}$$

We can write this as

$$T(a) = -i \sum_i c_i \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{(p+k)^2 - m_i^2}\right] \frac{1}{\left(k^2 - M_W^2\right)}$$
$$\times \frac{1}{\left[(k+q)^2 - M_W^2\right]} N^{\mu\nu} \Gamma_{\mu\nu}$$

where

$$c_i = \frac{g^2 e}{4} U_{ei}^* U_{\mu i} \tag{13.46}$$

and

$$N_{\mu\nu} = \bar{u}_e(p-q) \gamma_\mu \gamma \cdot (p+k) \gamma_\nu (1-\gamma_5) u_\mu(p). \tag{13.47}$$

Using the Feynman parameters, we can combine the denominators to get

$$\frac{1}{(p+k)^2 - m_i^2} \times \frac{1}{k^2 - M_W^2} \times \frac{1}{(k+q)^2 - M_W^2} = 2! \int \frac{d\alpha_1\, d\alpha_2\, \theta(1-\alpha_1-\alpha_2)}{A^3} \tag{13.48}$$

where

$$\begin{aligned} A &= \alpha_1[(p+k)^2 - m_i^2] + \alpha_2[(k+q)^2 - M_W^2] + (1-\alpha_1-\alpha_2)[k^2 - M_W^2] \\ &= (k + \alpha_1 p + \alpha_2 q)^2 - a^2 \end{aligned} \tag{13.49}$$

with

$$a^2 = \alpha_1 m_i^2 + (1-\alpha_1) M_W^2. \tag{13.50}$$

We have neglect $m_\mu^2$ as compared to $m_i^2$ or $M_W^2$. Collecting these factors we get

$$\begin{aligned} T(a) = i \sum_i c_i 2 \int d\alpha_1\, d\alpha_2\, \theta(1-\alpha_1-\alpha_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[(k+\alpha_1 p + \alpha_2 q)^2 - a^2\right]^3} S_1 \end{aligned}$$

where

$$S_1 = N_{\mu\upsilon}\Gamma^{\mu\nu}. \tag{13.51}$$

To simplify the integral, we can shift the integration variable, $k \to k - \alpha_1 p - \alpha_2 q$. Under this shift, we get for the $p \cdot \varepsilon$ term which contributes to $\mu \to e\gamma$, see CL-eqn (13.97),

$$S_1 \to \bar{S_1} = (p\cdot\varepsilon)[\bar{u}_e(1+\gamma_5)u_\mu]2m_\mu[2(1-\alpha_1)^2 + (2\alpha_1 - 1)\alpha_2]. \tag{13.52}$$

Momentum integration yields

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-a^2)^3} = \frac{(i)}{32\pi^2 a^2}. \tag{13.53}$$

The contribution to the invariant amplitude $A$ is then

$$A(a) = \sum_i c_i \frac{m_\mu}{16\pi^2} \int \frac{d\alpha_1\, d\alpha_2\, \theta(1-\alpha_1-\alpha_2)}{\alpha_1 m_i^2 + (1-\alpha_1)M_W^2} [2(1-\alpha_1)^2 + (2\alpha_1-1)\alpha_2]. \tag{13.54}$$

Integrating over $\alpha_2$, we get

$$A(a) = \sum_i c_i \frac{m_\mu}{16\pi^2} \left(\frac{1}{M_W^2}\right) \int \frac{d\alpha_1 (1-\alpha_1)^2 \left(\frac{3}{2} - \alpha_1\right)}{[(1-\alpha_1) + \alpha_1(m_i^2/M_W^2)]}. \tag{13.55}$$

From diagram (b) we have

$$\begin{aligned}
T_i(b) = {} & -i \sum_i \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p-q) \left(\frac{ig}{2\sqrt{2}}\right) U^*_{ei} \gamma_\lambda (1-\gamma_5) \frac{i}{\gamma \cdot (p+k) - m_i} \\
& \times \left(\frac{ig}{2\sqrt{2} M_W}\right) \frac{1}{2} U_{\mu i} [m_i(1-\gamma_5) - m_\mu(1+\gamma_5)] u_\mu(p) \frac{i}{\left(k^2 - M_W^2\right)} \\
& \times \frac{(-ig^{\lambda\nu})}{[(k+q)^2 - M_W^2]} (ieM_W \varepsilon_\nu) \\
= {} & i \sum_i c_i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p+k)^2 - m_i^2} \times \frac{1}{k^2 - M_W^2} \times \frac{1}{(k+q)^2 - M_W^2} \varepsilon^\lambda N_\lambda
\end{aligned} \tag{13.56}$$

where

$$N_\lambda = \bar{u}_e(p-q) \gamma_\lambda (1+\gamma_5) \left[m_i^2 - m_\mu(\gamma \cdot k + m_\mu)\right] u_\mu(p). \tag{13.57}$$

Combining denominator and shifting the integration variable, we have

$$T_i(b) = \sum_i 2ic_i \int d\alpha_1 \, d\alpha_2 \theta(1-\alpha_1-\alpha_2) \int \frac{d^4k}{(2\pi)^4} \frac{N_1}{(k^2-a^2)^3}. \tag{13.58}$$

Again, picking out the $p \cdot \varepsilon$ term,

$$N_1 = N_\lambda \varepsilon^\lambda \to N_1 = -2(p \cdot \varepsilon) \bar{u}_e(p-q)(1+\gamma_5) u_\mu(p) \alpha_2 m_\mu. \tag{13.59}$$

The combination to the invariant amplitude $A$ is then

$$\begin{aligned}
A(b) &= \sum_i c_i \frac{(-m_\mu)}{16\pi^2} \int \frac{d\alpha_1 \, d\alpha_2 \, \theta(1-\alpha_1-\alpha_2)\alpha_2}{\alpha_1 m_i^2 + (1-\alpha_2) M_W^2} \\
&= \sum_i c_i \frac{(-m_\mu)}{32\pi^2} \left(\frac{1}{M_W^2}\right) \int \frac{d\alpha_1 (1-\alpha_1)^2}{[(1-\alpha_1) + \alpha_1 (m_i^2/M_W^2)]}.
\end{aligned} \tag{13.60}$$

Diagram (c) gives the contribution

$$\begin{aligned}
T_i(c) = {} & -i \sum_i \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p-q) \left(\frac{ig}{2\sqrt{2} M_W}\right) U^*_{ei} [m_i(1+\gamma_5) - m_e(1-\gamma_5)] \\
& \times \frac{i}{\gamma \cdot (p+k) - m_i} \left(\frac{ig}{2\sqrt{2}}\right) \frac{1}{2} U_{\mu i} \gamma_\nu (1-\gamma_5) u_\mu(p) \frac{i}{\left(k^2 - M_W^2\right)} \\
& \times \frac{(-ig^{\lambda\nu})}{[(k+q)^2 - M_W^2]} (ieM_W \varepsilon_\lambda) \\
= {} & i \sum_i c_i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p+k)^2 - m_i^2} \times \frac{1}{k^2 - M_W^2} \times \frac{1}{(k+q)^2 - M_W^2} \varepsilon^\lambda N'_\lambda
\end{aligned} \tag{13.61}$$

where, after setting $m_e = 0$,

$$N'_\lambda = \bar{u}_e(p-q)\gamma_\lambda(1+\gamma_5)m_i^2 u_\mu(p). \tag{13.62}$$

It is easy to see that this does not contribute to the term $p \cdot \varepsilon$.

Diagram (d) is of the form

$$\begin{aligned}
T_i(d) = -i\sum_i \int \frac{d^4k}{(2\pi)^4}\bar{u}_e(p-q)\left(\frac{ig}{2\sqrt{2}M_W}\right)U^*_{ei}[m_i(1+\gamma_5) - m_e(1-\gamma_5)] \\
\times \frac{i}{\gamma\cdot(p+k) - m_i}\left(\frac{-ig}{2\sqrt{2}M_W}\right)U_{\mu i}[m_i(1-\gamma_5) - m_\mu(1+\gamma_5)]u_\mu(p) \\
\times \frac{i}{\left(k^2 - M_W^2\right)}\frac{i}{[(k+q)^2 - M_W^2]}(ie)\varepsilon\cdot(2k+q) \\
= -i\sum_i c_i \frac{m_i^2}{M_W^2}\int \frac{d^4k}{(2\pi)^4}[\bar{u}_e(p-q)(1+\gamma_5)u_\mu(p)]\frac{2k\cdot\varepsilon}{(p+k)^2 - m_i^2} \\
\times \frac{1}{k^2 - M_W^2} \times \frac{1}{(k+q)^2 - M_W^2} \\
= -4\sum_i c_i \frac{m_i^2}{M_W^2}(p\cdot\varepsilon)[\bar{u}_e(p-q)(1+\gamma_5)u_\mu(p)] \\
\times \int \frac{d\alpha_1\, d\alpha_2\, \theta(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)\alpha_1}{\alpha_1 m_i^2 + (1-\alpha_2)M_W^2}\left(\frac{-i}{32\pi^2}\right).
\end{aligned} \tag{13.63}$$

Thus we get for the invariant amplitude

$$\begin{aligned}
A(d) &= m_\mu \sum_i c_i \frac{(-1)}{32\pi^2}\left(\frac{m_i^2}{M_W^2}\right)\int \frac{d\alpha_1\, \alpha_1(1-\alpha_1)(1+\alpha_1)}{\alpha_1 m_i^2 + (1-\alpha_2)M_W^2} \\
&= m_\mu \sum_i c_i \frac{(-1)}{32\pi^2}\left(\frac{m_i^2}{M_W^4}\right)\int \frac{d\alpha_1\, \alpha_1(1-\alpha_1)(1+\alpha_1)}{\alpha_1(m_i^2/M_W^2) + (1-\alpha_1)}.
\end{aligned} \tag{13.64}$$

The total contribution is then

$$\begin{aligned}
A &= \sum_i [A_i(a) + A_i(b) + A_i(d)] \\
&= m_\mu \sum_i c_i \frac{1}{64\pi^2}\left(\frac{1}{M_W^2}\right)\int_0^1 \frac{d\alpha}{1-\alpha+\alpha(m_i^2/M_W^2)} \\
&\quad \times \left\{ -2(1-\alpha)^2(3-2\alpha) - 2(1-\alpha)^2 - 2\alpha(1-\alpha)(1+\alpha)\left(\frac{m_i^2}{M_W^2}\right)\right\} \\
&= m_\mu \sum_i c_i \frac{(-1)}{32\pi^2}\left(\frac{1}{M_W^2}\right) g\left(\frac{m_i^2}{M_W^2}\right)
\end{aligned} \tag{13.65}$$

where

$$g(x) = \int_0^1 \frac{(1-\alpha)d\alpha}{(1-\alpha)+\alpha x}[2(1-\alpha)(2-\alpha)+\alpha(1+\alpha)x] \tag{13.66}$$

and

$$c_i = \frac{g^2 e}{4} U_{ei}^* U_{\mu i}. \tag{13.67}$$

**(b)** The function $g(x)$ can be calculated as follows:

$$g(x) = \int_0^1 \frac{(1-\alpha)\,d\alpha}{(x-1)\,[\alpha+1/(x-1)]}[2(1-\alpha)(2-\alpha)+\alpha(1+\alpha)x]. \tag{13.68}$$

Let $y = 1/(x-1)$ or $x = (1+y)/y$.

$$\begin{aligned} g(x) &= \frac{1}{(x-1)} \int_0^1 \frac{(1-\alpha)\,d\alpha}{(\alpha+y)} \left[2(1-\alpha)(2-\alpha)+\alpha(1+\alpha)\frac{(1+y)}{y}\right] \\ &= \int_0^1 \frac{(1-\alpha)\,d\alpha}{(\alpha+y)}[2y(1-\alpha)(2-\alpha)+\alpha(1+\alpha)(1+y)] \\ &= \int_0^1 \frac{d\alpha}{(\alpha+y)}[-(1+3y)\alpha^3+8y\alpha^2+(1-9y)\alpha+4y]. \end{aligned} \tag{13.69}$$

To facilitate the integration we can write the numerator $d(\alpha) = -(1+3y)\alpha^3 + 8y\alpha^2 + (1-9y)\alpha + 4y$ as

$$d(\alpha) = d(\alpha) - d(-y) + d(-y) \tag{13.70}$$

where $d(-y) = 3y(1+y)^3$ so that

$$\begin{aligned} d(\alpha) - d(-y) = (\alpha+y)[&-(1+3y)\alpha^2 + 3y(y+3)\alpha \\ &+(-3y^3 - 9y^2 - 9y + 1)] \end{aligned} \tag{13.71}$$

and

$$\begin{aligned} g(x) = &\int_0^1 [-(1+3y)\alpha^2 + 3y(y+3)\alpha + (-3y^3 - 9y^2 - 9y + 1)]\,d\alpha \\ &+ d(-y)\int_0^1 \frac{d\alpha}{\alpha+y}. \end{aligned} \tag{13.72}$$

The integration brings about

$$\begin{aligned} \text{1st term} &= -(1+3y)\frac{1}{3} + 3y(y+3)\frac{1}{2} + (-3y^3 - 9y^2 - 9y + 1) \\ &= -3y^3 - \frac{15}{2}y^2 - \frac{11}{2}y + \frac{2}{3} \end{aligned} \tag{13.73}$$

and

$$\text{2nd term} = d(-y)\ln\left(\frac{1+y}{y}\right) = \frac{-3x^3}{(1-x)^4}\ln x. \tag{13.74}$$

Thus

$$g(x) = \left[\frac{3}{(1-x)^3} - \frac{15}{2}\left(\frac{1}{1-x}\right)^2 + \frac{11}{2}\left(\frac{1}{1-x}\right) + \frac{2}{3}\right] - \frac{3x^3}{(1-x)^4}\ln x.$$

For $x \ll 1$,

$$\begin{aligned} g(x) &= 3(1+3x) - \frac{15}{2}(1+2x) + \frac{11}{2}(1+x) + \frac{2}{3} + O(x^2) \\ &= \frac{5}{3} - \frac{x}{2} + O(x^2). \end{aligned} \tag{13.75}$$

In this way we see that for $m_i \ll M_W$, for all $i$, we get

$$\delta'_\nu = 2\sum_i U^*_{ei}U_{\mu i}\left[\frac{5}{3} - \frac{1}{2}\frac{m_i^2}{M_W^2}\right] = -\sum_i U^*_{ei}U_{\mu i}\left(\frac{m_i^2}{M_W^2}\right) \tag{13.76}$$

where we have used the unitary relation

$$\sum_i U^*_{ei}U_{\mu i} = 0. \tag{13.77}$$

This is the same as CL-eqn (13.113) in the text.

**(c)** But for $m_1, m_2 \ll M_W$ and $m_3 \gg M_W$, the situation is different:

$$\begin{aligned} \delta'_\nu &= 2\left(U^*_{e1}U_{\mu 1} + U^*_{e1}U_{\mu 1}\right)g(0) + 2U^*_{e3}U_{\mu 3}g(\infty) \\ &= 2U^*_{e3}U_{\mu 3}[g(\infty) - g(0)] = -2U^*_{e3}U_{\mu 3} \end{aligned} \tag{13.78}$$

because

$$g(0) = \frac{5}{3} \quad \text{and} \quad g(\infty) = \frac{2}{3}. \tag{13.79}$$

**Remark.** Since the GIM mechanism is not effective here, the branching ratio will be very large (compared to the experimental upper limit $< 10^{-10}$), if the mixing $U^*_{e3}U_{\mu 3}$ is not very small. Thus the coupling of electron or muon to any neutral lepton which is much heavier than a W boson must be highly suppressed.

## 13.5 Leptonic mixings in a vector-like theory

Consider a simple model of leptons, where there are two left-handed and two right-handed doublets and, in addition, there are two left-handed neutral leptons:

$$L_1 = \begin{pmatrix} n_e \\ e \end{pmatrix}_L, \quad L_2 = \begin{pmatrix} n_\mu \\ \mu \end{pmatrix}_L, \quad R_1 = \begin{pmatrix} n_e \\ e \end{pmatrix}_R, \quad R_2 = \begin{pmatrix} n_\mu \\ \mu \end{pmatrix}_R,$$
$$l_1 = n_{\sigma L}, \quad l_2 = n_{\tau L}.$$

Show that if the Higgs scalar is the usual $SU(2)\times U(1)$ doublet, then the weak eigenstates, $n_e, n_\mu, \ldots$, can be expressed in terms of mass eigenstates $N_1$, $N_2$, $\nu_3$, $\nu_4$ as follows:

$$\begin{aligned}(n_e)_R &= (\cos\phi N_1 + \sin\phi N_2)_R \\ (n_\mu)_R &= (-\sin\phi N_1 + \cos\phi N_2)_R\end{aligned} \tag{13.80}$$

$$\begin{aligned}(n_e)_L &= \left(\frac{m_e}{m_1}\cos\phi N_1 + \frac{m_e}{m_2}\sin\phi N_2 + U_{e3}\nu_3 + U_{e4}\nu_4\right)_L \\ (n_\mu)_L &= \left(-\frac{m_\mu}{m_1}\sin\phi N_1 + \frac{m_\mu}{m_2}\cos\phi N_2 + U_{\mu 3}\nu_3 + U_{\mu 4}\nu_4\right)_L\end{aligned} \tag{13.81}$$

where $U_{ai}$s are elements of the unitary matrix that diagonalize the mass matrix.

**Solution to Problem 13.5**

Because of the vector-like nature of the theory, we have the bare lepton mass term

$$-\mathcal{L}_0 = \sum_{i=1,2} m_i(\bar{L}_i R_i + h.c.) = m_e(\bar{e}e + \bar{n}_e n_e) + m_\mu(\bar{\mu}\mu + \bar{n}_\mu n_\mu) \tag{13.82}$$

and the mass terms arising from Yukawa couplings

$$\mathcal{L}_Y = m_{\sigma e}\bar{n}_{\sigma L} n_{eR} + m_{\tau e}\bar{n}_{\tau L} n_{eR} + m_{\sigma\mu}\bar{n}_{\sigma L} n_{\mu R} + m_{\tau\mu}\bar{n}_{\tau L} n_{\mu R}. \tag{13.83}$$

We can collect the mass terms of neutral leptons in the form

$$-\mathcal{L}_m = \bar{\psi}_{iR} M_{ij} \psi_{jL} + h.c. \tag{13.84}$$

where

$$\psi_{iR} = \begin{pmatrix} n_e \\ n_\mu \end{pmatrix}_R, \quad \psi_{jL} = \begin{pmatrix} n_e \\ n_\mu \\ n_\sigma \\ n_\tau \end{pmatrix}_L, \quad M_{ij} = \begin{pmatrix} m_e & 0 & m_{e\sigma} & m_{e\tau} \\ 0 & m_\mu & m_{\mu\sigma} & m_{\mu\tau} \end{pmatrix}.$$

To diagonalize this mass matrix, we use the biunitary transformation

$$V^\dagger M U = M_d, \qquad M_d = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}. \tag{13.85}$$

The mass eigenstates are

$$\left(\Psi_i'\right)_R = V_{ij}^\dagger \psi_{jR} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_R, \qquad \left(\Psi_i'\right)_L = \begin{pmatrix} N_1 \\ N_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}_L = U_{ij}^\dagger \psi_{jL}. \tag{13.86}$$

In general, the $2 \times 2$ unitary matrix $V$ can be taken to have the form

$$V = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{13.87}$$

From eqn (13.85)

$$V^{\dagger} M = M_d U^{\dagger} \tag{13.88}$$

we get

$$\begin{aligned}
&\begin{pmatrix} \cos\phi m_e & -\sin\phi m_\mu & \cdots & \cdots \\ \sin\phi m_e & \cos\phi m_\mu & \cdots & \cdots \end{pmatrix} \\
&= \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & \cdots \\ U_{\mu 1} & U_{\mu 2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\
&= \begin{pmatrix} m_1 U_{e1} & m_2 U_{\mu 2} & \cdots & \cdots \\ m_2 U_{e1} & m_2 U_{\mu 2} & \cdots & \cdots \end{pmatrix}.
\end{aligned} \tag{13.89}$$

Identifying matrix elements on both sides of this equation, we get (see Cheng and Li 1977 for more details)

$$\begin{aligned}
U_{e1} &= \frac{m_e}{m_1}\cos\phi, \qquad & U_{\mu 1} &= -\frac{m_\mu}{m_1}\sin\phi, \\
U_{e2} &= \frac{m_e}{m_2}\sin\phi, \qquad & U_{\mu 1} &= \frac{m_\mu}{m_2}\cos\phi.
\end{aligned} \tag{13.90}$$

## 13.6 Muonium-antimuonium transition

Compute the effective Lagrangian for the muonium–antimuonium transition $\mu^- + e^+ \to \mu^+ + e^-$ in the same model as the standard model but with massive neutrinos.

**Solution to Problem 13.6**

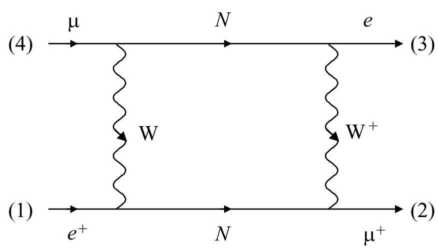



FIG. 13.3. The box diagram for muonium–antimuonium transition.

The only diagram contributing to this process is the box diagram. We are interested in the limit where all external momenta are small compared to $M_W$. In this approximation, the general box diagram with arbitrary masses for the internal fermion lines can be calculated in the 't Hooft–Feynman gauge

$$
\begin{aligned}
B(x,y) &= -i\left(\frac{ig}{\sqrt{2}}\right)^4 \int \frac{d^4k}{(2\pi)^4}[\bar{u}_L(4)\gamma^\nu(\gamma\cdot k + m_x)\gamma^\rho u_L(3)] \\
&\quad \times [\bar{u}_L(2)\gamma_\rho(\gamma\cdot k + m_y)\gamma_\nu u_L(1)] \\
&\quad \times \left(\frac{-i}{k^2 - M_W^2}\right)^2 \left(\frac{-i}{k^2 - m_x^2}\right)\left(\frac{-i}{k^2 - m_y^2}\right) \\
&= \left(\frac{-ig^4}{64\pi^3}\right)\int d^4k \left(\frac{k^2}{2}\right)\left(\frac{1}{k^2 - M_W^2}\right)^2 \left(\frac{1}{k^2 - m_x^2}\right)\left(\frac{1}{k^2 - m_y^2}\right) \\
&\quad \times \left[\bar{u}(4)\gamma^\nu\gamma^\lambda\gamma^\rho \tfrac{1}{2}(1-\gamma_5)u(3)\right]\left[\bar{u}(2)\gamma_\rho\gamma_\lambda\gamma_\nu \tfrac{1}{2}(1-\gamma_5)u(1)\right].
\end{aligned}
\tag{13.91}
$$

After making the Wick rotation the momentum integration can be reduced to a simple form that can be carried out explicitly:

$$
\begin{aligned}
&\int d^4k \left(\frac{k^2}{2}\right)\left(\frac{1}{k^2 - M_W^2}\right)^2 \left(\frac{1}{k^2 - m_x^2}\right)\left(\frac{1}{k^2 - m_y^2}\right) \\
&\quad = \frac{-i\pi^2}{4M_W^2}\int_0^\infty \frac{t^2\,dt}{(1+t)^2}\frac{1}{(t+x)}\frac{1}{(t+y)} = \frac{-i\pi^2}{4M_W^2} I(x,y)
\end{aligned}
\tag{13.92}
$$

where

$$
x = \frac{m_x^2}{M_W^2}, \qquad y = \frac{m_y^2}{M_W^2}. \tag{13.93}
$$

The function $I(x, y)$ is of the form

$$
I(x,y) = \frac{J(x) - J(y)}{x - y}, \qquad J(x) = \frac{1}{1-x} + \frac{x^2}{(1-x)^2}\ln x. \tag{13.94}
$$

The Dirac matrices can be simplified by the identity

$$
\gamma^\nu\gamma^\lambda\gamma^\rho = g^{\nu\lambda}\gamma^\rho + g^{\rho\lambda}\gamma^\nu - g^{\nu\rho}\gamma^\lambda - i\varepsilon^{\nu\lambda\rho\sigma}\gamma_5\gamma_\sigma. \tag{13.95}
$$

Thus we have

$$
\begin{aligned}
&\left[\bar{u}(4)\gamma^\nu\gamma^\lambda\gamma^\rho \tfrac{1}{2}(1-\gamma_5)u(3)\right]\left[\bar{u}(2)\gamma_\rho\gamma_\lambda\gamma_\nu \tfrac{1}{2}(1-\gamma_5)u(1)\right] \\
&\quad = 10\left[\bar{u}(4)\gamma^\lambda \tfrac{1}{2}(1-\gamma_5)u(3)\right]\left[\bar{u}(2)\gamma_\lambda \tfrac{1}{2}(1-\gamma_5)u(1)\right] \\
&\qquad + \varepsilon^{\nu\lambda\rho\sigma}\varepsilon_{\rho\lambda\nu\tau}\left[\bar{u}(4)\gamma_5\gamma_\sigma \tfrac{1}{2}(1-\gamma_5)u(3)\right]\left[\bar{u}(2)\gamma_5\gamma^\tau \tfrac{1}{2}(1-\gamma_5)u(1)\right]
\end{aligned}
\tag{13.96}
$$

Using

$$\varepsilon^{\nu\lambda\rho\sigma}\varepsilon_{\rho\lambda\nu\tau} = -6g^{\sigma}_{\tau}, \qquad \gamma_5(1-\gamma_5) = -(1-\gamma_5) \tag{13.97}$$

we get for the box diagram

$$B(x,y) = -g^4 \frac{1}{64\pi^2 M_W^2} I(x,y)\left[\bar{u}(4)\gamma^{\lambda}\tfrac{1}{2}(1-\gamma_5)u(3)\right] \times \left[\bar{u}(2)\gamma_{\lambda}\tfrac{1}{2}(1-\gamma_5)u(1)\right].$$

The effective interaction is then of the form

$$\mathcal{L}_{eff}(\mu^- + e^+ \to \mu^+ + e^-) = g_{eff}\left[\bar{e}\gamma^{\lambda}\tfrac{1}{2}(1-\gamma_5)\mu\right]\left[\bar{e}\gamma_{\lambda}\tfrac{1}{2}(1-\gamma_5)\mu\right] \tag{13.98}$$

where

$$g_{eff} = \frac{G_F^2}{16\pi^2}\sum_{i,j}\left(U_{\mu i}U_{ei}^* U_{\mu j}U_{ej}^*\right)\left[\frac{J(x_i)-J(x_j)}{x_i - x_j}\right] \tag{13.99}$$

with $x_i = m_i^2/M_W^2$.

For the case $x \ll 1$,

$$J(x) = \frac{1}{1-x} + \frac{x^2}{(1-x)^2}\ln x \to 1 + x + x^2 + x^2 \ln x$$

$$\frac{J(x)-J(y)}{x-y} = 1 + (x+y) + \frac{x^2\ln x - y^2 \ln y}{x-y}. \tag{13.100}$$

Then we get for the effective coupling constant

$$g_{eff} = \frac{G_F^2}{16\pi^2}\sum_{i,j}\left(U_{\mu i}U_{ei}^* U_{\mu j}U_{ej}^*\right)\left[(x_i + x_j) + \frac{x_i^2 \ln x_i - x_j^2 \ln x_j}{x_i - x_j}\right]. \tag{13.101}$$

# 14 Grand unification

## 14.1 Content of SU(5) representations

The $SU(3) \times SU(2)$ content of the SU(5) representation **5** is given by

$$\mathbf{5} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}). \tag{14.1}$$

Show that the SU(5) antisymmetric tensor representation **10** has the following decomposition

$$\mathbf{10} = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}) \tag{14.2}$$

and the adjoint representation **24** has

$$\mathbf{24} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{3}^*, \mathbf{2}). \tag{14.3}$$

Also, find the decomposition of the symmetric tensor representation **15**.

**Solution to Problem 14.1**

From $\mathbf{5} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$ we can form the second rank antisymmetric tensor:

$$\begin{aligned}(\mathbf{5} \times \mathbf{5})_{(-)} &= ([(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})] \times [(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})])_{(-)} \\ &= ((\mathbf{3} \times \mathbf{3})_{(-)}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, (\mathbf{2} \times \mathbf{2})_{(-)})\end{aligned} \tag{14.4}$$

where the subscript $_{(-)}$ denotes antisymmetrization (while $_{(+)}$ will be used to denote symmetrization). In SU(3), we have $(\mathbf{3} \times \mathbf{3})_{(-)} = \mathbf{3}^*$, and in SU(2), $(\mathbf{2} \times \mathbf{2})_{(-)} = \mathbf{1}$. Then we get

$$(\mathbf{5} \times \mathbf{5})_{(-)} = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}). \tag{14.5}$$

On the other hand, as representations in SU(5), we have

$$(\mathbf{5} \times \mathbf{5})_{(-)} = \mathbf{10} \qquad (\mathbf{5} \times \mathbf{5})_{(+)} = \mathbf{15}^*. \tag{14.6}$$

Thus we get

$$\mathbf{10} = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}), \tag{14.7}$$

all of which corresponds to known particles in the standard model. Similarly, we can work out the symmetric part

$$\begin{aligned}(\mathbf{5} \times \mathbf{5})_{(+)} &= ([(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})] \times [(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})])_{(+)} \\ &= ((\mathbf{3} \times \mathbf{3})_{(+)}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, (\mathbf{2} \times \mathbf{2})_{(+)}) \\ &= (\mathbf{6}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}).\end{aligned} \tag{14.8}$$

Namely, the single SU(5) representation having exactly the same number of states (with the correct quantum numbers) as one generation of standard model fermions clearly does not correspond to any particles we have observed so far:

$$\mathbf{15}^* = (\mathbf{6}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}). \tag{14.9}$$

The adjoint representation can be obtained by the product of the fundamental representation and its conjugate:

$$\mathbf{5} \times \mathbf{5}^* = \mathbf{24} + \mathbf{1} \tag{14.10}$$

To work out the SU(3) × SU(2) decomposition we note that

$$\begin{aligned}\mathbf{5} \times \mathbf{5}^* &= [(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})] \times [(\mathbf{3}^*, \mathbf{1}) + (\mathbf{1}, \mathbf{2})] \\ &= (\mathbf{3} \times \mathbf{3}^*, \mathbf{1}) + (\mathbf{1}, \mathbf{2} \times \mathbf{2}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{3}^*, \mathbf{2}) \\ &= (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{3}^*, \mathbf{2}).\end{aligned} \tag{14.11}$$

Subtracting $(\mathbf{1}, \mathbf{1})$ from both sides we get

$$\mathbf{24} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{3}^*, \mathbf{2}). \tag{14.12}$$

## 14.2 Higgs potential for SU(5) adjoint scalars

In the Higgs sector of the SU(5) model, if we neglect Higgs in representation **5**, we can write the scalar potential in the form

$$V(H) = -m^2 Tr(H^2) + \lambda_1 (Tr(H^2))^2 + \lambda_2 Tr(H^4) \tag{14.13}$$

where $H$ is the Higgs field in the adjoint representation of SU(5) and is represented as a $5 \times 5$ hermitian traceless matrix. Here, for simplicity, we have imposed a symmetry of $H \to -H$ to remove the cubic term.

**(a)** Show that $H$ can be transformed into a real diagonal traceless matrix

$$H = U H_d U^\dagger \qquad \text{with} \quad H_d = \begin{pmatrix} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ & & & h_4 & \\ & & & & h_5 \end{pmatrix} \tag{14.14}$$

where $h_1 + h_2 + h_3 + h_4 + h_5 = 0$.

**(b)** Show that at the minimum, the diagonal elements $h_i$s can take at most three different values. From this result, discuss the most general form of symmetry breakings that can be induced by a **24** adjoint Higgs field.

**Solution to Problem 14.2**

**(a)** The adjoint representation $H$ has the following SU(5) transformation property:

$$H \to H' = UHU^\dagger. \tag{14.15}$$

Since any hermitian matrix can be diagonalized by a unitary matrix, we can choose $U$ such that $H$ is the unitary equivalent to a real diagonal matrix:

$$H_d = UHU^\dagger = \begin{pmatrix} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ & & & h_4 & \\ & & & & h_5 \end{pmatrix}. \tag{14.16}$$

The trace is invariant under unitary transformation, $TrH = 0$, which implies that

$$h_1 + h_2 + h_3 + h_4 + h_5 = 0. \tag{14.17}$$

**(b)** With $H$ in the diagonal form, the scalar potential is simplified:

$$V(H) = -m^2 \sum_i h_i^2 + \lambda_1 \left( \sum_i h_i^2 \right)^2 + \lambda_2 \sum_i h_i^4. \tag{14.18}$$

Since $h_i$s are not all independent, we need to use the Lagrange multiplier $\mu$ to account for the constraint $\sum_i h_i = 0$. Write

$$\begin{aligned} V' &= V(H) - \mu Tr(H) \\ &= -m^2 \sum_i h_i^2 + \lambda_1 \left( \sum_i h_i^2 \right)^2 + \lambda_2 \sum_i h_i^4 - \mu \sum_i h_i. \end{aligned} \tag{14.19}$$

Then

$$\frac{\partial V'}{\partial h_i} = -2m^2 h_i + 4\lambda_1 \left( \sum_j h_j^2 \right) h_i + 4\lambda_2 h_i^3 - \mu = 0. \tag{14.20}$$

Thus at the minimum all $h_i$s satisfy the same cubic equation

$$4\lambda_2 x^3 + 4\lambda_1 a x - 2m^2 x - \mu = 0 \qquad \text{with} \quad a = \sum_j h_j^2. \tag{14.21}$$

This means that $h_i$s can at most take on three different values, $\phi_1, \phi_2$, and $\phi_3$, which are the three roots of the cubic equation. Note that the absence of the $x^2$ term in the cubic equation implies that

$$\phi_1 + \phi_2 + \phi_3 = 0. \tag{14.22}$$

Let $n_1, n_2, n_3$ be the number of times $\phi_1, \phi_2, \phi_3$ appear in $H_d$,

$$H_d = \begin{pmatrix} \phi_1 & & & & & \\ & \ddots & & & & \\ & & \phi_2 & & & \\ & & & \ddots & & \\ & & & & \phi_3 & \\ & & & & & \ddots \end{pmatrix} \qquad \text{with} \quad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0. \tag{14.23}$$

Thus $H_d$ is invariant under $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{SU}(n_3)$ transformations. This implies that the most general form of symmetry breaking is $\mathrm{SU}(n) \to \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{SU}(n_3)$, as well as additional U(1) factors which leave $H_d$ invariant.

**Remark.** To find the absolute minimum we need to use the relations

$$n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0, \qquad \phi_1 + \phi_2 + \phi_3 = 0 \tag{14.24}$$

to compare different choices of $\{n_1, n_2, n_3\}$ to get the one with smallest $V(H)$. It turns out that for the case of interest there are two possible patterns for the symmetry breaking,

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \tag{14.25}$$

or

$$\mathrm{SU}(5) \to \mathrm{SU}(4) \times \mathrm{U}(1) \tag{14.26}$$

depending on the relative magnitudes of the parameters, $\lambda_1$ and $\lambda_2$.

## 14.3 Massive gauge bosons in SU(5)

For the adjoint representation $H$, written as a $5 \times 5$ traceless hermitian matrix, construct the covariant derivative $D_\mu H$. Calculate the mass spectra of the gauge bosons from the covariant derivative if the vacuum expectation value is given by

$$\langle H \rangle_0 = v \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}. \tag{14.27}$$

**Solution to Problem 14.3**

In the SU($n$) group we have the following transformation for the fundamental representation $\psi_j$ with $j = 1, 2, \ldots, n$:

$$\psi_j \to \psi_j' = U_j^k \psi_k = \left(\delta_j^k + i\varepsilon_j^k\right)\psi_k \tag{14.28}$$

where we have also written out the form for an infinitesimal transformation. The conjugate representation transforms differently. For $\psi^j \equiv (\psi_j)^*$ we have

$$\psi^j \to \psi'^j = \left(\delta^j_k - i\varepsilon^j_k\right)\psi^k. \tag{14.29}$$

The adjoint representation $H^k_j$ transforms in the same way as the product $\psi_j\psi^k$:

$$H'^k_j = \left(\delta^l_j + i\varepsilon^l_j\right)\left(\delta^k_m - i\varepsilon^k_m\right)H^m_l = H^k_j + i\varepsilon^l_j H^k_l - i\varepsilon^k_m H^m_j. \tag{14.30}$$

The covariant derivative is obtained by the replacement of $\varepsilon^k_m \to gW^k_m$ in the above expression:

$$(D_\mu H)^k_j = \partial_\mu H^k_j + ig(W_\mu)^l_j H^k_l - ig(W_\mu)^k_m H^m_j. \tag{14.31}$$

Or in terms of matrix multiplication

$$D_\mu H = \partial_\mu H + ig(W_\mu H - HW_\mu) = \partial_\mu H + ig[W_\mu, H] \tag{14.32}$$

where $W_\mu, H$ are $5 \times 5$ traceless hermitian matrices. The gauge boson masses come from the covariant derivatives

$$\mathcal{L}_W = Tr\left[D_\mu\langle H\rangle(D_\mu\langle H\rangle)^\dagger\right] \to g^2 Tr([W_\mu, \langle H\rangle][W^\mu, \langle H\rangle]). \tag{14.33}$$

It is easy to see that

$$[W_\mu, \langle H\rangle]^j_k = (W_\mu)^j_k(H_k - H_j) \tag{14.34}$$

where

$$\langle H\rangle^j_k = H_k\delta^j_k \quad \text{(no sum)}. \tag{14.35}$$

Equation (14.34) implies that if $H_k = H_j$, the gauge field $(W_\mu)^j_k$ is massless, and if $H_k \neq H_j$, then the corresponding gauge field is massive. From the VEV given by

$$\langle H\rangle_0 = v\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{14.36}$$

the gauge boson fields $(W_\mu)^j_i$ having $i = 1, 2, 3$ and $j = 4, 5$ are massive, $M^2 = 25g^2v^2$, while $i, j = 1, 2, 3$ and $i, j = 4, 5$ are still massless. In other words, the symmetry-breaking pattern is given by,

$$\text{SU}(5) \to \text{SU}(3) \times \text{SU}(2) \times \text{U}(1). \tag{14.37}$$

The remanent U(1) corresponds to a generator which has the same diagonal form as that given in eqn (14.36). The number of massive gauge bosons is then

$$24 - (8 + 3 + 1) = 12. \tag{14.38}$$

## 14.4 Baryon number non-conserving operators

Write down all possible dimension-6 operators which are invariant under the standard model group $SU(3)_C \times SU(2)_L \times U(1)$ and violate the baryon number $(B)$ conservation.

**Solution to Problem 14.4**

In order to form a colour singlet state out of quark fields we need $\bar{q}q$ or $qqq$. But the quark–antiquark combination does not violate baryon number conservation. Hence we are interested in composite operators that contain three quarks in the form $q_\alpha q_\beta q_\gamma \varepsilon^{\alpha\beta\gamma}$ or equivalently $\bar{q}^c_\alpha q_\beta \bar{q}^c_\gamma \varepsilon^{\alpha\beta\gamma}$, where $\alpha$, $\beta$, $\gamma$ are the colour indices. Thus these dimension-6 operators have the generic structure $(\bar{q}^c q)(\bar{q}^c l)$. As for the $SU(2)_L$ symmetry, it is clear that there are three possibilities: 4 doublets, 2 doublets, or all singlets.

For simplicity, we take only one generation of fermions. SU(2) indices are written in the Latin alphabet.

(i) 4 doublets

$$
\begin{aligned}
O^{(1)} &= \left(\bar{q}^c_{i\alpha L} q_{j\beta L}\right)\left(\bar{q}^c_{k\gamma L} l_{nL}\right)\varepsilon_{\alpha\beta\gamma}\varepsilon_{ij}\varepsilon_{kn} \\
O^{(2)} &= \left(\bar{q}^c_{i\alpha L} q_{j\beta L}\right)\left(\bar{q}^c_{k\gamma L} l_{nL}\right)\varepsilon_{\alpha\beta\gamma}\,(\boldsymbol{\tau}\varepsilon)_{ij}\cdot(\boldsymbol{\tau}\varepsilon)_{kn}
\end{aligned}
\tag{14.39}
$$

where $q_{1\alpha L} = u_{\alpha L}$ and $q_{2\alpha L} = d_{\alpha L}$.

(ii) 2 doublets

$$
\begin{aligned}
O^{(3)} &= \left(\bar{d}^c_{\alpha R} u_{\beta R}\right)\left(\bar{q}^c_{i\gamma L} l_{jL}\right)\varepsilon_{\alpha\beta\gamma}\varepsilon_{ij} \\
O^{(4)} &= \left(\bar{q}^c_{i\alpha L} q_{j\beta L}\right)\left(\bar{u}^c_{\gamma R} l_{aR}\right)\varepsilon_{\alpha\beta\gamma}\varepsilon_{ij}.
\end{aligned}
\tag{14.40}
$$

(iii) 4 singlets

$$
\begin{aligned}
O^{(5)} &= \left(\bar{d}^c_{\alpha R} u_{\beta R}\right)\left(\bar{u}^c_{\gamma R} l_{aR}\right)\varepsilon_{\alpha\beta\gamma} \\
O^{(6)} &= \left(\bar{u}^c_{\alpha R} u_{\beta R}\right)\left(\bar{d}^c_{\gamma R} l_{aR}\right)\varepsilon_{\alpha\beta\gamma}.
\end{aligned}
\tag{14.41}
$$

## 14.5 SO($n$) group algebra

Consider a real $n$-dimensional space with vector $\mathbf{x} = (x_1, \ldots, x_n)$. A rotation in this space can be represented as

$$x_i \to x'_i = R_{ij} x_j \tag{14.42}$$

where $R$ is an $n \times n$ orthogonal matrix, $R^T R = R R^T = 1$.

**(a)** For infinitesimal rotation, show that $R_{ij}$ can be written as

$$R_{ij} = \delta_{ij} + \varepsilon_{ij} \qquad \text{with} \quad \varepsilon_{ij} = -\varepsilon_{ji}. \tag{14.43}$$

**(b)** For any function of **x**, this infinitesimal rotation induces a transformation which can be written as

$$f(\mathbf{x}) \to f(\mathbf{x}') = f(\mathbf{x}) + \frac{i\varepsilon_{ij}}{2} J_{ij} f(\mathbf{x}). \tag{14.44}$$

Show that the operators $J_{ij}$ can be written as

$$J_{ij} = -i(x_i \partial_j - x_j \partial_i) \qquad i, j = 1, 2, \ldots, n. \tag{14.45}$$

**(c)** Show that $J_{ij}$s satisfy the commutation relations,

$$[J_{ij}, J_{kl}] = i(\delta_{lk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk}). \tag{14.46}$$

**(d)** Show that in the group SO($n$) with either even $n = 2m$ or odd $n = 2m + 1$, we can find $m$ mutually commuting generators.

**(e)** For the simple case of $n = 3$, if we define

$$J_i = \tfrac{1}{2}\varepsilon_{ijk} J_{jk} \tag{14.47}$$

then the commutators in (c) reduce to the usual angular momentum algebra

$$[J_i, J_j] = i\varepsilon_{ijk} J_k. \tag{14.48}$$

**(f)** For $n = 4$, define $K_i = J_{i4}$, show that

$$[K_i, K_j] = i\varepsilon_{ijk} J_k \qquad \text{and} \qquad [J_i, K_j] = i\varepsilon_{ijk} K_k \tag{14.49}$$

where $J_i$s are defined in part (e). Also if we define

$$A_i = \tfrac{1}{2}(J_i + K_i), \qquad B_i = \tfrac{1}{2}(J_i - K_i) \tag{14.50}$$

show that

$$[A_i, A_j] = i\varepsilon_{ijk} A_k, \quad [B_i, B_j] = i\varepsilon_{ijk} B_k, \quad [A_i, B_j] = 0. \tag{14.51}$$

### Solution to Problem 14.5

**(a)** Write the matrix equation $RR^T = 1$ in components, we have $R_{ij} R_{ik} = \delta_{jk}$. For $R_{ij} = \delta_{ij} + \varepsilon_{ij}$, with $\varepsilon_{ij} \ll 1$, it becomes

$$(\delta_{ij} + \varepsilon_{ij})(\delta_{ik} + \varepsilon_{ik}) = \delta_{jk} \;\;\Rightarrow\;\; \varepsilon_{jk} = -\varepsilon_{kj}. \tag{14.52}$$

Thus we have $\frac{1}{2}n(n-1)$ independent parameters for the orthogonal matrices $R$.

**(b)**
$$x_i \to x_i' = R_{ij} x_j = x_i + \varepsilon_{ij} x_j. \tag{14.53}$$

Then

$$\begin{aligned} f(x_i') &= f(x_i + \varepsilon_{ij} x_j) = f(x_i) + \varepsilon_{ij} x_j \frac{\partial f}{\partial x_i} \\ &= f(x_i) + \frac{\varepsilon_{ij}}{2}\left(x_j \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial x_j}\right). \end{aligned} \tag{14.54}$$

If we write the left-hand side as

$$f(x_i') = f(x_i) + \tfrac{1}{2} i\varepsilon_{ij} J_{ij} f(x_i) \tag{14.55}$$

we get

$$J_{ij} = -i(x_i\partial_j - x_j\partial_i). \tag{14.56}$$

**(c)** From the simple commutator

$$[\partial_i, x_j] = \delta_{ij}, \tag{14.57}$$

we get

$$[x_i\partial_j, x_k\partial_l] = x_i\delta_{jk}\partial_l - x_k\delta_{il}\partial_j. \tag{14.58}$$

From this it is straightforward to get the commutators for $J_{ij}$s:

$$[J_{ij}, J_{kl}] = -i(\delta_{jk}J_{il} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk}). \tag{14.59}$$

**(d)** If indices $(i, j, k, l)$ are all different, the commutator is zero. Thus the following generators will commute with each other:

$$\begin{aligned} &\{J_{12}, J_{34}, J_{56}, \ldots, J_{n-1,n}\} && \text{for } n \text{ even} \\ &\{J_{12}, J_{34}, J_{56}, \ldots, J_{n-2,n-1}\} && \text{for } n \text{ odd.} \end{aligned} \tag{14.60}$$

**Remark.** This set of $n/2$ (or $(n-1)/2$ for odd $n$) mutually commuting generators is said to form the *Cartan subalgebra*.

**(e)** To recover the familiar angular momentum commutation relations from eqn (14.59), we use the identification

$$J_1 = J_{23}, \quad J_2 = J_{31}, \quad J_3 = J_{12} \tag{14.61}$$

$$[J_1, J_2] = [J_{23}, J_{31}] = -iJ_{21} = iJ_3. \tag{14.62}$$

Similarly, we can obtain

$$[J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2. \tag{14.63}$$

**(f)** From $K_i = J_{i4}$, we get, from eqn (14.59),

$$[K_1, K_2] = [J_{14}, J_{24}] = -i(-J_{12}) = iJ_3. \tag{14.64}$$

Similarly,

$$[K_2, K_3] = iJ_1, \quad [K_3, K_1] = iJ_2, \quad \text{or} \quad [K_i, K_j] = i\varepsilon_{ijk}J_k. \tag{14.65}$$

For the other commutators, we have

$$[J_1, K_2] = [J_{23}, J_{24}] = -i(-J_{34}) = iK_3. \tag{14.66}$$

Similarly,

$$[J_2, K_3] = iK_1, \quad [J_3, K_1] = iK_2, \quad \text{or} \quad [J_i, K_j] = i\varepsilon_{ijk}K_k. \tag{14.67}$$

These commutators can be simplified by defining

$$A_i = \tfrac{1}{2}(J_i + K_i), \quad B_i = \tfrac{1}{2}(J_i - K_i) \tag{14.68}$$

which gives

$$[A_i, B_j] = \tfrac{1}{4}[J_i + K_i, J_j - K_j] = \tfrac{1}{4}i\varepsilon_{ijk}(J_k + K_k - J_k - K_k) = 0$$

$$[A_i, A_j] = \tfrac{1}{4}[J_i + K_i, J_j + K_j] = \tfrac{1}{4}i\varepsilon_{ijk}(J_k + K_k + J_k + K_k) = i\varepsilon_{ijk}A_k.$$

Similarly,

$$[B_i, B_j] = i\varepsilon_{ijk}B_k. \tag{14.69}$$

This means that SO(4) algebra is isomorphic to SU(2) × SU(2), generated by $A_i$s and $B_j$s separately. Also this implies that the SO(4) group contains three distinct SU(2) subgroups, namely those formed by $\{A_i\}$, $\{B_i\}$, or $\{J_i\}$ generators.

## 14.6 Spinor representations of SO($n$)

Consider the $n$-dimensional real space with coordinates $(x_1, x_2, \ldots, x_n)$.

**(a)** Show that if we write the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$ as the square of a linear form

$$x_1^2 + x_2^2 + \cdots + x_n^2 = (x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n)^2 \tag{14.70}$$

then the coefficient $\gamma$s satisfy the anticommutation relation

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{14.71}$$

This is usually referred to as the *Clifford algebra*.

**(b)** Show that if we take $\gamma_i$s to be hermitian matrices, then $\gamma_i$s have to be even-dimensional matrices.

**(c)** For the even case, $n = 2m$, show that the following set of matrices satisfies the anticommutation relations in (14.71):

$$m = 1, \qquad \gamma_1^{(1)} = \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2^{(1)} = \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{14.72}$$

and for the iteration from $m$ to $m+1$ we have

$$\gamma_i^{(m+1)} = \begin{pmatrix} \gamma_i^{(m)} & 0 \\ 0 & -\gamma_i^{(m)} \end{pmatrix}, \qquad i = 1, 2, 3, \ldots, 2m \tag{14.73}$$

$$\gamma_{2m+1}^{(m+1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_{2m+2}^{(m+1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{14.74}$$

**(d)** Consider a rotation in space $(x_1, \ldots, x_n)$

$$x_i \to x_i{}' = O_{ik} x_k \tag{14.75}$$

which induces a transformation on the $\gamma_i$s

$$\gamma_i{}' = O_{ik}\gamma_k. \tag{14.76}$$

Show that $\{\gamma_i'\}$ satisfy the same anticommutation relations (14.71):

$$\{\gamma_i', \gamma_j'\} = 2\delta_{ij} \tag{14.77}$$

**(e)** Because the original $\{\gamma_i\}$ form a complete set of matrix algebra, they are related to the new $\{\gamma_i'\}$ by a similarity transformation

$$\gamma_i{}' = S(O)\gamma_i S^{-1}(O) \qquad \text{or} \qquad O_{ik}\gamma_k = S(O)\gamma_i S^{-1}(O) \tag{14.78}$$

where $S(O)$ is some non-singular $2^m \times 2^m$ matrix. If we write these transformations in the infinitesimal form

$$O_{ik} = \delta_{ik} + \varepsilon_{ik}, \qquad S(O) = 1 + \tfrac{1}{2} i S^{ij}\varepsilon_{ij}, \qquad \text{with} \quad \varepsilon_{ik} = -\varepsilon_{ki},$$

show that

$$i[S_{ij}, \gamma_k] = (\delta_{ik}\gamma_j - \delta_{jk}\gamma_i) \tag{14.79}$$

and that such $S_{ij}$ can be related to the $\gamma$ matrices by

$$S_{kl} = \frac{i}{2}\sigma_{kl} \equiv \frac{i}{4}\left[\gamma_k, \gamma_l\right]. \tag{14.80}$$

**(f)** Show that for the matrices given in part (c), we have

$$\sigma_{ij}^{(m+1)} = \begin{pmatrix} \sigma_{ij}^{(m)} & 0 \\ 0 & \sigma_{ij}^{(m)} \end{pmatrix}, \qquad i, j = 1, 2, 3, \ldots, 2m \tag{14.81}$$

$$\begin{gathered} \sigma_{i,2m+1}^{(m+1)} = i\begin{pmatrix} 0 & \gamma_i^{(m)} \\ -\gamma_i^{(m)} & 0 \end{pmatrix} \qquad \sigma_{i,2m+2}^{(m+1)} = i\begin{pmatrix} 0 & \gamma_i^{(m)} \\ -\gamma_i^{(m)} & 0 \end{pmatrix} \\ \sigma_{2m+1,2m+2}^{(m+1)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{14.82}$$

**(g)** Show that $S_{kl} = \frac{i}{4}[\gamma_k, \gamma_l]$, as given in part (e), satisfies the commutation relation for SO($n$), as given in Problem 14.5:

$$[S_{ij}, S_{kl}] = -i(\delta_{jk}S_{il} + \delta_{il}S_{jk} - \delta_{ik}S_{jl} - \delta_{jl}S_{ik}). \tag{14.83}$$

$\{S_{ij}\}$ then form the spinor representation of the SO($n$) group. Clearly it is a $2^m = 2^{n/2}$ dimensional representation. (In Problem 14.8, we shall study its decomposition into two sets of $2^{m-1}$ spinor states.)

**Solution to Problem 14.6**

**(a)** To solve for $\gamma$s from the equation $\sum_i x_i^2 = (x_1\gamma_1 + \cdots + x_n\gamma_n)^2$ it is clear that $\gamma_i$s cannot be the usual real or complex numbers. The simplest possibility is that $\gamma_i$s are hermitian matrices so that $\gamma_i\gamma_j \neq \gamma_j\gamma_i$:

$$\sum_i x_i^2 = \left(\sum_i x_i\gamma_i\right)^2 = \sum_{i,j} x_i x_j \gamma_i\gamma_j = \sum_{i,j} x_i x_j \tfrac{1}{2}(\gamma_i\gamma_j + \gamma_j\gamma_i). \tag{14.84}$$

Thus we need to have

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}. \tag{14.85}$$

**(b)** From eqn (14.85) we see that, because $\gamma_j^2 = 1$,

$$\gamma_j(\gamma_i\gamma_j + \gamma_j\gamma_i) = 2\gamma_j \qquad \text{or} \qquad \gamma_j\gamma_i\gamma_j = \gamma_i, \qquad \text{no sum on } j. \tag{14.86}$$

Taking the trace of this final relation we get

$$Tr(\gamma_j\gamma_i\gamma_j) = Tr(\gamma_i). \tag{14.87}$$

But for the case $i \neq j$, eqn (14.85) implies that

$$Tr(\gamma_j\gamma_i\gamma_j) = Tr(-\gamma_i\gamma_j\gamma_j) = Tr(-\gamma_i). \tag{14.88}$$

Thus combining eqns (14.87) and (14.88) we have

$$Tr(\gamma_i) = 0. \tag{14.89}$$

On the other hand, $\gamma_i^2 = 1$ implies that the eigenvalues of $\gamma_i$ are either $+1$ or $-1$. This means that to get $Tr(\gamma_i) = 0$, the numbers of $+1$ and $-1$ eigenvalues have to be the same. Thus $\gamma_i$ must be even-dimensional matrices.

**(c) The $m = 1$ case** Because the $2 \times 2$ Pauli matrices

$$\{\tau_i, \tau_j\} = 2\delta_{ij} \tag{14.90}$$

satisfy the anticommutation relation of (14.85), we can choose $\gamma_i^{(1)} = \tau_i$ with $i = 1, 2$.

**The $m > 1$ case** Using the recursion relation (14.73) we get ($i, j = 1, 2, 3, \ldots, 2m$)

$$\begin{aligned}\left\{\gamma_i^{(m+1)}, \gamma_j^{(m+1)}\right\} &= \begin{pmatrix} \left\{\gamma_i^{(m)}, \gamma_j^{(m)}\right\} & 0 \\ 0 & \left\{\gamma_i^{(m)}, \gamma_j^{(m)}\right\} \end{pmatrix} \\ &= \begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & 2\delta_{ij} \end{pmatrix} = 2\delta_{ij}\end{aligned} \tag{14.91}$$

$$\left\{\gamma_i^{(m+1)}, \gamma_{2m+1}^{(m+1)}\right\} = \begin{pmatrix} 0 & \gamma_i^{(m)} \\ -\gamma_i^{(m)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\gamma_i^{(m)} \\ \gamma_i^{(m)} & 0 \end{pmatrix} = 0, \tag{14.92}$$

$$\left(\gamma_{2m+1}^{(m+1)}\right)^2 = 1. \tag{14.93}$$

Similarly,

$$\left\{\gamma_i^{(m+1)}, \gamma_{2m+2}^{(m+1)}\right\} = 0, \qquad \left\{\gamma_{2m+1}^{(m+1)}, \gamma_{2m+2}^{(m+1)}\right\} = 0, \qquad \left(\gamma_{2m+2}^{(m+1)}\right)^2 = 1. \tag{14.94}$$

**(d)** To compute the anticommutator of the transformed gamma matrices:

$$(\gamma_i'\gamma_j' + \gamma_j'\gamma_i') = O_{ik}O_{jl}(\gamma_k\gamma_l + \gamma_l\gamma_k) = O_{ik}O_{jl}2\delta_{kl} = 2\delta_{ij} \tag{14.95}$$

where we have used the fact that $O$ is an orthogonal matrix.

**(e)** From $O_{ik}O_{jk} = \delta_{ij}$, we get for $O_{ij} = \delta_{ij} + \varepsilon_{ij}$ with $\varepsilon_{ij} \ll 1$

$$(\delta_{ik} + \varepsilon_{ik})(\delta_{jk} + \varepsilon_{jk}) = \delta_{ij} \qquad \text{or} \qquad \varepsilon_{ij} = -\varepsilon_{ji}. \tag{14.96}$$

Thus if we write $S(O) = 1 + \frac{1}{2}iS_{ij}\varepsilon_{ij}$, then $S_{ij}$ is also antisymmetric in $i \leftrightarrow j$. From $O_{ik}\gamma_k = S(O)\gamma_i S^{-1}(O)$, we get

$$(\delta_{ik} + \varepsilon_{ik})\gamma_k = \left(1 + \tfrac{1}{2}iS_{ab}\varepsilon_{ab}\right)\gamma_i\left(1 - \tfrac{1}{2}iS_{kl}\varepsilon_{kl}\right) \tag{14.97}$$

or

$$\gamma_i + \varepsilon_{ik}\gamma_k = \gamma_i + i\frac{\varepsilon_{kl}}{2}[S_{kl}, \gamma_i]. \tag{14.98}$$

Write

$$\varepsilon_{ik}\gamma_k = \varepsilon_{lk}\gamma_k\delta_{il} = \tfrac{1}{2}\varepsilon_{lk}(\gamma_k\delta_{il} - \gamma_l\delta_{ik}). \tag{14.99}$$

Thus we get

$$i[S_{kl}, \gamma_j] = (\gamma_l\delta_{jk} - \gamma_k\delta_{jl}). \tag{14.100}$$

We now show that

$$S_{kl} = \frac{i}{2}\sigma_{kl} \equiv \frac{i}{4}[\gamma_k, \gamma_l] \tag{14.101}$$

will satisfy the commutation relation (14.100). Since $k \neq l$, in $S_{kl}$, the above relation can be written

$$S_{kl} = \frac{i}{4}(\gamma_k\gamma_l - \gamma_l\gamma_k) = \frac{i}{2}\gamma_k\gamma_l. \tag{14.102}$$

Then

$$\begin{aligned} i[S_{kl}, \gamma_i] &= -\tfrac{1}{2}[\gamma_k\gamma_l, \gamma_i] \\ &= -\tfrac{1}{2}(\gamma_k\{\gamma_l, \gamma_i\} - \{\gamma_k, \gamma_i\}\gamma_l) = (\gamma_l\delta_{ik} - \gamma_k\delta_{il}). \end{aligned} \tag{14.103}$$

**(f)** From the definition of $\sigma_{ij}$ and the recursion relation (14.73), we have for the case of $n = m + 1$

$$\begin{aligned}
\sigma_{ij}^{(m+1)} &= \frac{1}{2}\left[\gamma_i^{(m+1)}, \gamma_j^{(m+1)}\right] \\
&= \begin{pmatrix} \frac{1}{2}\left[\gamma_i^{(m)}, \gamma_j^{(m)}\right] & 0 \\ 0 & \frac{1}{2}\left[-\gamma_i^{(m)}, -\gamma_j^{(m)}\right] \end{pmatrix} \\
&= \begin{pmatrix} \sigma_{ij}^{(m)} & 0 \\ 0 & \sigma_{ij}^{(m)}. \end{pmatrix}
\end{aligned} \tag{14.104}$$

The remaining results stated in part (f ) can be demonstrated in a similar manner.

**(g)** From the relation (14.102), the commutator can be evaluated:

$$\begin{aligned}
[S_{ij}, S_{kl}] &= i\tfrac{1}{2}[S_{ij}, \gamma_k\gamma_l] = i\tfrac{1}{2}([S_{ij}, \gamma_k]\gamma_l + \gamma_k[S_{ij}, \gamma_l]) \\
&= \tfrac{1}{2}((\gamma_j\delta_{ik} - \gamma_i\delta_{jk})\gamma_l + \gamma_k(\gamma_j\delta_{il} - \gamma_i\delta_{jl})) \\
&= -i(\delta_{jk}S_{il} + \delta_{li}S_{jk} - \delta_{ik}S_{jl} - \delta_{jl}S_{ik}).
\end{aligned} \tag{14.105}$$

## 14.7 Relation between SO(2$n$) and SU($n$) groups

The U($n$) group consists of transformations that act on the $n$-dimensional complex vectors, leaving their scalar product $(w \cdot z) = \sum_i w_i^* z_i$ invariant.

**(a)** Show that the SU($n$) transformations which leave Re$(w \cdot z)$ invariant can be identified as those in an SO(2$n$) group. Thus, the SU($n$) is a subgroup of SO(2$n$).

**(b)** If we write the SO(2$n$) matrix in the form $R = e^M$, where $M$ is an antisymmetric $2n \times 2n$ matrix in the form

$$M = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} \tag{14.106}$$

where $A, C$ are antisymmetric matrices and $B$ is an arbitrary $n \times n$ matrix, show that $R = e^M$ also belongs to U($n$) if $M$ has the specific form

$$M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad \text{with } A \text{ antisymmetric and } B \text{ symmetric.} \tag{14.107}$$

**(c)** The $2n$-dimensional representation of SO(2$n$) decomposes as $\mathbf{n} + \mathbf{n}^*$ under SU($n$). In other words, if we write the $2n$-dimensional real vector in the form

$$\mathbf{r} = (a_1, \ldots, a_n, b_1, \ldots, b_n) \equiv (a_i, b_j), \tag{14.108}$$

show that, for the unitary matrices written in the form as given in (b), the combination $a_j + ib_j$ transformed into themselves and so did the combination $a_j - ib_j$.

**(d)** Work out the decomposition of the adjoint representation of SO(2$n$) in terms of the irreducible representations of SU($n$).

**Solution to Problem 14.7**

**(a)** In the scalar product $(w \cdot z) = \sum_i w_i^* z_i$ we can write $w_i$ and $z_i$ in terms of their real and imaginary parts

$$w_j = a_j + i b_j, \qquad z_j = a'_j + i b'_j. \tag{14.109}$$

The scalar product can then be put in the form

$$(w \cdot z) = \sum_{j=1}^{n} \left(a_j a'_j + b_j b'_j\right) + i \sum_{j=1}^{n} \left(a_j b'_j - b_j a'_j\right) \tag{14.110}$$

which gives

$$\mathrm{Re}(w \cdot z) = \sum_{j=1}^{n} (a_j a'_j + b_j b'_j) \tag{14.111}$$

If we collect $a_j$ and $b_j$ to define a $2n$-dimensional real vector $\mathbf{r} = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ then $\mathrm{Re}(w \cdot z)$ can be written as the scalar product of $2n$-dimensional real vectors

$$\mathbf{r} \cdot \mathbf{r}' = \sum_{j=1}^{n} (a_j a'_j + b_j b'_j). \tag{14.112}$$

The transformations which leaves this scalar product invariant are just the SO$(2n)$ transformations

$$r_i \to r'_i = R_{ij} r_j \qquad \text{where} \quad R R^T = R^T R = 1. \tag{14.113}$$

From this we see that the SU$(n)$ group is a subgroup of SO$(2n)$ whose transformations leaves both $\mathrm{Re}(w \cdot z)$ and $\mathrm{Im}(w \cdot z)$ invariant.

**(b) and (c)** By definition a U$(n)$ transformation on the $n$-dimensional complex vector space is of the form

$$z_i \to z'_i = U_{ij} z_j \qquad \text{with} \quad U U^\dagger = U^\dagger U = 1 \tag{14.114}$$

where the unitary matrix $U$ can be written as

$$U = e^H \qquad \text{with } H = -H^\dagger \text{ being an antihermitian matrix.} \tag{14.115}$$

For infinitesimal $H$, we have $U \simeq 1 + H$ and

$$z'_i = z_i + H_{ij} z_j \;\; \Rightarrow \;\; z'^*_i = z^*_i + H^*_{ij} z^*_j. \tag{14.116}$$

Thus

$$z'_i + z'^*_i = (z_i + z^*_i) + \tfrac{1}{2}(H_{ij} + H^*_{ij})(z_j + z^*_j) + \tfrac{1}{2}(H_{ij} - H^*_{ij})(z_j - z^*_j) \tag{14.117}$$

and

$$a_i' = a_i + A_{ij}a_j + B_{ij}b_j \tag{14.118}$$

where $a$ and $b$ are the real and imaginary parts of $z$:

$$\begin{aligned} a_i &= \tfrac{1}{2}(z_i + z_i^*), \qquad & b_i &= -i\tfrac{1}{2}(z_i - z_i^*), \\ A_{ij} &= \tfrac{1}{2}(H_{ij} + H_{ij}^*), \qquad & B_{ij} &= i\tfrac{1}{2}(H_{ij} - H_{ij}^*) \end{aligned} \tag{14.119}$$

are all real. Since $H$ is antihermitian, we have

$$H_{ij}^* = -H_{ji}. \tag{14.120}$$

This implies that

$$A_{ij} = -A_{ji} \qquad \text{and} \qquad B_{ij} = B_{ji}. \tag{14.121}$$

Similarly,

$$b_i' = b_i - B_{ij}a_j + A_{ij}b_j. \tag{14.122}$$

We can combine these two transformations as

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + M \begin{pmatrix} a \\ b \end{pmatrix} \tag{14.123}$$

where $M$ is an antisymmetric matrix of a specific form

$$M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad \text{with} \quad A = -A^T, \; B = B^T. \tag{14.124}$$

**(d)** From (b) and (c) we have learned that the vector in SO($2n$) decomposes into $\mathbf{n} + \mathbf{n}^*$ of SU($n$). The generators in SO($2n$) can be associated with second-rank anti-symmetric tensors. This implies the decomposition of SO($2n$) generators as

$$\begin{aligned} [(\mathbf{n}+\mathbf{n}^*)\times(\mathbf{n}+\mathbf{n}^*)]_{(-)} &= (\mathbf{n}\times\mathbf{n})_{(-)} + (\mathbf{n}\times\mathbf{n}^*) + (\mathbf{n}^*\times\mathbf{n}^*)_{(-)} \\ &= \frac{\mathbf{n(n-1)}}{\mathbf{2}} \oplus (\mathbf{n}^2 - \mathbf{1}) \oplus \mathbf{1} \oplus \frac{\mathbf{n(n-1)}^*}{\mathbf{2}} \end{aligned}$$

where the subscript $_{(-)}$ means antisymmetrization. For example, we can decompose the **45** generators of SO(10) in terms of the irreducible representations of SU(5):

$$\mathbf{45} = \mathbf{10} + \mathbf{24} + \mathbf{1} + \mathbf{10}^*. \tag{14.125}$$

## 14.8 Construction of SO($2n$) spinors

The $\gamma$-matrices given in Problem 14.6 can also be written as a tensor product in the form (Note: the integer $m$ of Problem 14.6 is being called $n$ in this problem)

$$\begin{aligned} \gamma_i^{(n+1)} &= \gamma_i^{(n)} \times \tau_3, \qquad & i &= 1, 2, \ldots, 2n, \\ \gamma_{2n+1}^{(n+1)} &= 1^{(n)} \times \tau_1, \qquad & \gamma_{2n+2}^{(n+1)} &= 1^{(n)} \times \tau_2 \end{aligned}$$

where $1^{(n)}$ and $\gamma_i^{(n)}$s are $2^n \times 2^n$ hermitian matrices.

**(a)** Show that

$$\gamma_{2k}^{(n)} = 1 \times 1 \times \cdots \times \tau_2 \times \tau_3 \times \cdots \tau_3$$
$$\gamma_{2k-1}^{(n)} = 1 \times 1 \times \cdots \times \tau_1 \times \tau_3 \times \cdots \tau_3 \qquad (14.126)$$

where the $2 \times 2$ identity matrix 1 appears $(k-1)$ times and $\tau_3$ appears $(n-k)$ times and

$$\sigma_{2k-1,2k}^{(n)} = 1 \times 1 \times \cdots \times \tau_3 \times 1 \times \cdots 1. \qquad (14.127)$$

**(b)** For the chirality matrix, we define

$$\gamma_{FIVE} = (-i)^n (\gamma_1 \gamma_2 \cdots \gamma_{2n}). \qquad (14.128)$$

Show that $\gamma_{FIVE}$ can be written as the direct product of $n$ Pauli $\tau_3$ matrices:

$$\gamma_{FIVE} = \tau_3 \times \tau_3 \times \cdots \tau_3 \qquad (14.129)$$

and

$$\gamma_{FIVE}^{(n+1)} = \begin{pmatrix} \gamma_{FIVE}^{(n)} & 0 \\ 0 & -\gamma_{FIVE}^{(n)} \end{pmatrix}. \qquad (14.130)$$

**(c)** Since the natural basis for Pauli matrices are spin-up $|+\rangle$ and spin-down $|-\rangle$ states, we can take as the basis for the $\gamma$-matrices the tensor product of such states

$$|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\rangle \equiv |\varepsilon_1\rangle|\varepsilon_2\rangle \ldots |\varepsilon_n\rangle \qquad \text{with} \quad \varepsilon_i = \pm 1. \qquad (14.131)$$

Show that $2^n$ such states in the SO($2n$) spinor representation decompose irreducibly into two set of $2^{n-1}$ states, called $S^+$ and $S^-$. They have the property of

$$\prod_{i=1}^{n} \varepsilon_i = \begin{cases} +1 \\ -1 \end{cases} \quad \text{for} \quad \begin{matrix} S^+ \\ S^- \end{matrix} \text{ states.} \qquad (14.132)$$

**(d)** For the case of $n = 2$, suppose we embed the SU(2) group into SO(4) by identifying $\tau_k$ of SU(2) generators with $B_k$ generators of SO(4) as defined in part (f) of Problem 14.5. Show that spinor representation $S^+$ and $S^-$ reduce with respect to the subgroup SU(2) as

$$S^+ \to \mathbf{1} + \mathbf{1}, \qquad S^- \to \mathbf{2}. \qquad (14.133)$$

**(e)** Show that for the case $n = 3$, the spinor representation $S^+$ and $S^-$ of SO(6) reduced with respect to the subgroup SU(3) as

$$S^+ \to \mathbf{3}^* + \mathbf{1}, \qquad S^- \to \mathbf{3} + \mathbf{1}. \qquad (14.134)$$

**(f)** Show that under SU(5), the spinor representation $S^+ \sim \mathbf{16}$ of SO(10) reduces as $\mathbf{16} = \mathbf{10} + \mathbf{5} + \mathbf{1}$.

For more details of SO($2n$) spinor construction, see Wilczek and Zee (1982).

**Solution to Problem 14.8**

**(a) and (b)** Let us work out the tensor product expression of a few low-dimensional $\gamma^{(n)}$ matrices:

For the $n = 1$ case:

$$\gamma_1^{(1)} = \tau_1, \qquad \gamma_2^{(1)} = \tau_2, \qquad \gamma_{FIVE}^{(1)} = -i\tau_1\tau_2 = \tau_3. \tag{14.135}$$

For the $n = 2$ case:

$$\begin{aligned}
\gamma_1^{(2)} &= \tau_1 \times \tau_3, \qquad \gamma_2^{(2)} = \tau_2 \times \tau_3,\\
\gamma_3^{(2)} &= 1 \times \tau_1, \qquad \gamma_4^{(2)} = 1 \times \tau_2,\\
\gamma_{FIVE}^{(2)} &= (-i)^2\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}\gamma_4^{(2)} = -(\tau_1\tau_2) \times (\tau_3\tau_1\tau_2)\\
&= \tau_3 \times \tau_3.
\end{aligned} \tag{14.136}$$

For the $n = 3$ case:

$$\begin{aligned}
\gamma_1^{(3)} &= \tau_1 \times \tau_3 \times \tau_3, \qquad \gamma_2^{(3)} = \tau_2 \times \tau_3 \times \tau_3,\\
\gamma_3^{(3)} &= 1 \times \tau_1 \times \tau_3, \qquad \gamma_4^{(3)} = 1 \times \tau_2 \times \tau_3,\\
\gamma_5^{(3)} &= 1 \times 1 \times \tau_1, \qquad \gamma_6^{(3)} = 1 \times 1 \times \tau_2,\\
\gamma_{FIVE}^{(3)} &= (-i)^3\gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)}\gamma_4^{(3)}\gamma_5^{(3)}\gamma_6^{(3)}\\
&= \tau_3 \times \tau_3 \times \tau_3.
\end{aligned} \tag{14.137}$$

From these it is not hard to obtain the general $(n)$ case by induction

$$\gamma_{2k-1}^{(n)} = 1 \times 1 \times \cdots \tau_1 \times \tau_3 \times \cdots \times \tau_3, \quad \gamma_{2k}^{(n)} = 1 \times 1 \times \cdots \tau_2 \times \tau_3 \times \cdots \times \tau_3,$$

where the identity matrix 1 appears $(k-1)$ times and $\tau_3$ appears $(n-k)$ times. We can also explicitly calculate the commutators:

$$\sigma_{2k-1,2k}^{(n)} = \frac{i}{2}\left[\gamma_{2k-1}^{(n)}, \gamma_{2k}^{(n)}\right] = 1 \times 1 \times \cdots \tau_3 \times 1 \times \cdots \times 1 \tag{14.138}$$

and deduce

$$\gamma_{FIVE}^{(n)} = \tau_3 \times \tau_3 \times \tau_3 \times \cdots \times \tau_3. \tag{14.139}$$

**(c)** It is easy to see that $\gamma_{FIVE}^{(n)}$ anticommutes with $\gamma$s. For example,

$$\gamma_{FIVE}^{(n)}\gamma_{2k-1} = \tau_3 \times \tau_3 \times \cdots \times (\tau_3\tau_1) \times \tau_3^2 \cdots \times \tau_3^2 \tag{14.140}$$

$$\gamma_{2k-1}\gamma_{FIVE}^{(n)} = \tau_3 \times \tau_3 \times \cdots \times (\tau_1\tau_3) \times \tau_3^2 \cdots \times \tau_3^2 \tag{14.141}$$

which gives

$$\left\{\gamma_{FIVE}^{(n)}, \gamma_{2k-1}\right\} = 0. \tag{14.142}$$

Since $\sigma_{ij}$ is quadratic in $\gamma_i$s,

$$\left[\gamma_{FIVE}^{(n)}, \sigma_{ij}\right] = 0 \tag{14.143}$$

which implies that $\gamma_{FIVE}^{(n)}$ commutes with the generator $J_{ij}$, which are represented by $J_{ij} = \frac{1}{2}\sigma_{ij}$ in the spinor representation. From $\left(\gamma_{FIVE}^{(n)}\right)^2 = 1$, we can decompose the $2^n$-dimensional spinor into two $2^{n-1}$-dimensional representations $S^{\pm}$ having $\gamma_{FIVE}^{(n)}$ eigenvalues of $\pm 1$ (Schur's lemma). It is clear that

$$\begin{aligned}\gamma_{FIVE}^{(n)}|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\rangle &= (\tau_3 \times \tau_3 \times \cdots \times \tau_3)|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\rangle \\ &= \left(\prod_{i=1}^{n} \varepsilon_i\right)|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\rangle. \end{aligned} \tag{14.144}$$

Thus $S^+$ states have $\prod_{i=1}^n \varepsilon_i = 1$ and $S^-$ states $\prod_{i=1}^n \varepsilon_i = -1$.

**(d)** Applying the above results to the SO(4) group which has the irreducible spinors

$$S^+ : \mathbf{2}^+ \sim \begin{pmatrix} |++\rangle \\ |--\rangle \end{pmatrix} \qquad \text{and} \qquad S^- : \mathbf{2}^- \sim \begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix} \tag{14.145}$$

and embedding SU(2) into SO(4), we can identify the SU(2) generators with a subset in the SO(4):

$$\text{`}\tau_k\text{'} \to B_k = J_k - K_k = \tfrac{1}{2}\varepsilon_{kij}J_{ij} - J_{k4}. \tag{14.146}$$

For example,

$$\text{`}\tau_3\text{'} \to J_{12} - J_{34} \to \tfrac{1}{2}(\sigma_{12} - \sigma_{34}) = \tfrac{1}{2}(-\tau_3 \times 1 + 1 \times \tau_3). \tag{14.147}$$

In $S^+$, we have $|++\rangle$, $|--\rangle$ and both have a zero '$\tau_3$' eigenvalue, e.g.

$$\text{`}\tau_3\text{'}|++\rangle = \tfrac{1}{2}[(-1)\cdot 1 + 1 \cdot (+1)]|++\rangle = 0 \tag{14.148}$$

Similarly, for the $|--\rangle$ state

$$\text{`}\tau_3\text{'}\begin{pmatrix} |++\rangle \\ |--\rangle \end{pmatrix} = 0. \tag{14.149}$$

Namely, both members of the $S^+$ spinor transform trivially under SU(2). This implies that $\mathbf{2}^+ \to \mathbf{1} + \mathbf{1}$ under SU(2). For the $S^-$ states it can be similarly worked out and the result can be written as

$$\text{`}\tau_3\text{'}\begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix}. \tag{14.150}$$

Thus we have $\mathbf{2}^- \to \mathbf{2}$. Namely, the SO(4) spinor $S^-$ is simply an SU(2) spinor.

**Remark.** The identification of the SU(2) generators $T_{\hat{b}}^{\hat{a}}$ with those in the SO(4) can also be carried out through the identification of their respective indices: $\hat{a} = \hat{1}, \hat{2}$ for SU(2) and $i = 1, 2, 3, 4$ for SO(4).

$$\hat{1} = 1 + i2, \qquad \hat{2} = 3 + i4$$

$$\begin{aligned} T_{\hat{2}}^{\hat{2}} = J_{3+i4,3-i4} = iJ_{43} - iJ_{34} = -2iJ_{34} &= -\tfrac{1}{2}\sigma_{34} \\ &= -\tfrac{1}{2}(1 \times \tau_3) \end{aligned} \tag{14.151}$$

Similarly, we have $T_{\hat{1}}^{\hat{1}} = -\frac{1}{2}(\tau_3 \times 1)$, so that we check with the above result,

$$\text{`}\tau_3\text{'} = T_{\hat{1}}^{\hat{1}} - T_{\hat{2}}^{\hat{2}} = \tfrac{1}{2}(-\tau_3 \times 1 + 1 \times \tau_3). \tag{14.152}$$

**(e)** We can embed the SU(3) group into SO(6) with the identification

$$\hat{1} = 1 + i2, \qquad \hat{2} = 3 + i4, \qquad \hat{3} = 5 + i6, \qquad \text{with} \quad \hat{\alpha} = \text{SU(3) index}.$$

The SU(3) generators $W_{\hat{\alpha}}^{\hat{\beta}}$ are related to $J_{\hat{\alpha}\hat{\beta}^{\text{s}}}$ by these relation of indices, e.g.

$$W_{\hat{1}}^{\hat{1}} = J_{1+i2,1-i2} = iJ_{21} - iJ_{12} = -2iJ_{12} = -\tfrac{1}{2}\sigma_{12} = -\tfrac{1}{2}(\tau_3 \times 1 \times 1). \tag{14.153}$$

Similarly,

$$W_{\hat{2}}^{\hat{2}} = -\tfrac{1}{2}(1 \times \tau_3 \times 1), \qquad W_{\hat{3}}^{\hat{3}} = -\tfrac{1}{2}\,(1 \times 1 \times \tau_3)\,. \tag{14.154}$$

For $S^+$, we have the states $|{+}{+}{+}\rangle$, $|{+}{-}{-}\rangle$, $|{-}{+}{-}\rangle$, $|{-}{-}{+}\rangle$, and their quantum numbers are given by

| State | $W_1^1 - W_2^2$ | $W_1^1 + W_2^2 - 2W_3^3$ |
|---|---|---|
| $\|{+}{+}{+}\rangle$ | $0$ | $0$ |
| $\|{+}{-}{-}\rangle$ | $-\frac{1}{2}$ | $-1$ |
| $\|{-}{+}{-}\rangle$ | $+\frac{1}{2}$ | $-1$ |
| $\|{-}{-}{+}\rangle$ | $0$ | $2$ |

(It may be helpful to think of $W_1^1 - W_2^2$ as the third component of isospin and $W_1^1 + W_2^2 - 2W_3^3$ as the hypercharge operator.) Thus we see that the states $|{+}{-}{-}\rangle$, $|{-}{+}{-}\rangle$, $|{-}{-}{+}\rangle$ form the triplet $\mathbf{3}^*$ representation under SU(3)

$$\begin{pmatrix} |{+}{-}{-}\rangle \\ |{-}{+}{-}\rangle \\ |{-}{-}{+}\rangle \end{pmatrix} \sim \mathbf{3}^*. \tag{14.155}$$

This means that $S^+$ decomposes under SU(3) as $\mathbf{4}^+ \to \mathbf{3}^* + \mathbf{1}$. Similarly, $S^-$ decomposes as $\mathbf{4}^- \to \mathbf{3} + \mathbf{1}$.

**(f)** In the spinor representation $S^+$ for SO(10), we can denote the states by

$$|\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\rangle \qquad \text{with} \quad \prod_i \varepsilon_i = 1. \tag{14.156}$$

We can identify the first two $\varepsilon_i$s as the spinor states of SO(4) which contains $\text{SU(2)}_L$ and the last three $\varepsilon_i$s with SO(6) which contains $\text{SU(3)}_C$. Then the $\text{SU(2)}_L \times \text{SU(3)}_C$ quantum numbers of spinor representation $S^+$ are given by

$$\begin{pmatrix} |{+}{+}\rangle \\ |{-}{-}\rangle \end{pmatrix} \times \begin{pmatrix} |{+}{-}{-}\rangle \\ |{-}{+}{-}\rangle \\ |{-}{-}{+}\rangle \end{pmatrix} \to 2(\mathbf{1}, \mathbf{3}^*) \to u_R, d_R \tag{14.157}$$

$$\begin{pmatrix} |{+}{+}\rangle \\ |{-}{-}\rangle \end{pmatrix} \times |{+}{+}{+}\rangle \to 2(\mathbf{1}, \mathbf{1}) \to \nu_R, e_R \tag{14.158}$$

$$\begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix} \times |+++\rangle \to (\mathbf{2},\mathbf{1}) \to \begin{pmatrix} \nu \\ e \end{pmatrix}_L \tag{14.159}$$

$$\begin{pmatrix} |+-\rangle \\ |-+\rangle \end{pmatrix} \times \begin{pmatrix} |+--\rangle \\ |-+-\rangle \\ |--+\rangle \end{pmatrix} \to (\mathbf{2},\mathbf{3}) \to \begin{pmatrix} u \\ d \end{pmatrix}_L . \tag{14.160}$$

It is easy to see that these are just $\mathbf{5} + \mathbf{10}^* + \mathbf{1}$ representations of SU(5), see Problem 14.1. The SU(5) singlet can be identified with the right-handed neutrino $\nu_R$.

# 15 Magnetic monopoles

## 15.1 The Sine–Gordon equation

Consider the Lagrangian for a scalar field $\phi$ in two-dimensions (respectively, one space and one time)

$$\mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{a}{b}[1 - \cos(b\phi)] \tag{15.1}$$

where $a$ and $b$ are constants.

**(a)** Show that the equation of motion for this Lagrangian is of the form

$$\partial_0^2\phi - \partial_x^2\phi + a\sin(b\phi) = 0 \tag{15.2}$$

which is called the *Sine–Gordon equation*.

**(b)** Verify that the field configuration

$$\phi(x,t) = \frac{4}{b}\tan^{-1}\left\{\exp\left[\pm(ab)^{1/2}\frac{(x - vt)}{(1 - v^2)^{1/2}}\right]\right\}, \tag{15.3}$$

with $v$ being an arbitrary constant, is a solution of the Sine–Gordon equation.

**(c)** Show that the effective potential $V(\phi)$, given by $V(\phi) = a(1 - \cos b\phi)/b$, has degenerate minima at

$$\phi_n^{(\min)} = n\frac{2\pi}{b} \tag{15.4}$$

where $n$ is an integer, and the field configuration, given by eqn (15.3 ), interpolates between two such minima of $V(\phi)$:

$$\begin{aligned} \phi(x = -\infty, t) &= \phi_n^{(\min)} \\ \phi(x = +\infty, t) &= \phi_{n+1}^{(\min)}. \end{aligned} \tag{15.5}$$

**(d)** Show that the energy carried by the configuration (15.3) at $t = 0$ is

$$E = 8\left(\frac{a}{b^3}\right)^{1/2}. \tag{15.6}$$

**(e)** If we write the Lagrangian $\mathcal{L}$ in powers of $\phi$, find the mass and the quartic coupling constant in terms of $a$ and $b$. Express the energy $E = 8(a/b^3)^{1/2}$ in terms of the mass and coupling constant.

**Solution to Problem 15.1**

**(a)** The equation of motion which follows from

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial^{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \tag{15.7}$$

is the Sine–Gordon equation:

$$\partial^{\mu}\partial_{\mu}\phi + a \sin b\phi = 0, \tag{15.8}$$

i.e.

$$\partial_0^2\phi - \partial_x^2\phi + a \sin b\phi = 0. \tag{15.9}$$

**(b)** We will show that the field of the form

$$\phi(x) = \frac{4}{b} \tan^{-1} \xi \tag{15.10}$$

where

$$\xi = \exp\left[(ab)^{1/2}\gamma(x - vt)\right], \tag{15.11}$$

and $\gamma = (1 - v^2)^{-1/2}$ with $v$ being arbitrary, satisfies eqn (15.9).

Using the formula

$$\frac{\partial}{\partial x} \tan^{-1} y = \frac{1}{1 + y^2} \frac{\partial y}{\partial x} \tag{15.12}$$

we obtain

$$\frac{\partial \phi}{\partial x} = \frac{4}{b} \frac{1}{1 + \xi^2} \frac{\partial \xi}{\partial x} = \frac{4}{b} \frac{\xi}{1 + \xi^2} (ab)^{1/2}\gamma \tag{15.13}$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{4a\gamma^2\xi(1 - \xi^2)}{(1 + \xi^2)^2}. \tag{15.14}$$

Because the function $\phi(x, t)$ has the space–time dependence through the combination of $(x - vt)$, the second derivatives must be related (as in the conventional wave equation) $\partial_0^2\phi - v^2\partial_x^2\phi = 0$. The above result can then be written as the first and second terms of the Sine–Gordon eqn (15.9):

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{-4a\xi(1 - \xi^2)}{(1 + \xi^2)^2}. \tag{15.15}$$

Now calculate the third term in the Sine–Gordon equation:

$$\sin b\phi = \sin 4(\tan^{-1} \xi) = \sin 4\theta \qquad \text{with} \quad \theta = \tan^{-1} \xi. \tag{15.16}$$

With the help of the formula $\sin 4\theta = 2 \sin 2\theta \cos 2\theta$, and

$$\sin 2\theta = \frac{2 \tan\theta}{1+\tan^2\theta} = \frac{2\xi}{1+\xi^2} \qquad \text{and} \qquad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\xi^2}{1+\xi^2}, \tag{15.17}$$

the sine term can be evaluated,

$$a \sin b\phi = a \sin 4\theta = \frac{4\xi(1-\xi^2)a}{(1+\xi^2)^2} \tag{15.18}$$

which just cancel the first two terms calculated in eqn (15.15):

$$\partial_0^2\phi - \partial_x^2\phi + a \sin b\phi = 0. \tag{15.19}$$

**Remark.** This solution has the space–time dependence through the variable $\xi = \exp\left[(ab)^{1/2}\gamma(x - vt)\right]$. One simple way to understand this is to note that if we write

$$x' = \frac{x - vt}{(1-v^2)^{1/2}} \tag{15.20}$$

then $x'$ is related to $x$ by a Lorentz transformation. Thus in this variable $x'$, so that $\xi = \exp[(ab)^{1/2}x']$, the function $\phi(\xi)$ is expected to be a solution to the *time-independent* equation of

$$\partial_x^2\phi = a \sin b\phi. \tag{15.21}$$

Let us demonstrate this. To solve this static equation by integration, we can multiply both sides by $\partial_x\phi$ so that it becomes

$$\frac{1}{2}\frac{d}{dx}\left(\frac{d\phi}{dx}\right)^2 = -\frac{a}{b}\frac{d}{dx}(\cos b\phi) \tag{15.22}$$

or

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{a}{b}(\cos b\phi) = c. \tag{15.23}$$

If we use the boundary condition of $\partial_x\phi = 0$ and $b\phi = 2n\pi$, as $x \to \pm\infty$, then the integration constant is fixed to be $c = a/b$. We have

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{2a}{b}(1 - \cos b\phi) = \frac{4a}{b}\sin^2\frac{b\phi}{2} \tag{15.24}$$

or

$$\frac{d\phi}{dx} = \pm 2\left(\frac{a}{b}\right)^{1/2}\sin\frac{b\phi}{2}. \tag{15.25}$$

Integrate over the equation

$$\pm\int \frac{d\phi}{\sin(b\phi/2)} = 2\left(\frac{a}{b}\right)^{1/2}\int dx \tag{15.26}$$

or

$$\pm(ab)^{1/2}x' = \frac{1}{2}\ln\left|\frac{1-\cos(b\phi/2)}{1+\cos(b\phi/2)}\right| = \ln\left(\tan\frac{b\phi}{4}\right) \tag{15.27}$$

which just checks with our result of

$$\phi = \frac{4}{b}\tan^{-1}\left\{\exp\left[\pm(ab)^{1/2}x'\right]\right\}. \tag{15.28}$$

Thus we can get the general solution by boosting this static solution in the $x$-direction by a velocity $v$. In other words, the general solution is moving in the $x$-direction with a velocity $v$.

**(c)** From $V(\phi) = a(1-\cos b\phi)/b$ we have

$$\frac{\partial V}{\partial\phi} = a\sin b\phi, \tag{15.29}$$

so that $\partial V/\partial\phi = 0$ can be satisfied by $b\phi = m\pi$ with $m = 0, \pm1, \pm2, \ldots$. Such extremum points are minima if

$$\frac{\partial^2 V}{\partial\phi^2} = ab\cos b\phi \geq 0. \tag{15.30}$$

Thus if we take $ab \geq 0$, then $\cos m\pi \geq 0$, only for even $m = 2n$. Thus the minima of $V(\phi)$ are located at

$$\phi_n^{(\min)} = \frac{2n\pi}{b} \qquad n = 0, \pm1, \pm2, \ldots. \tag{15.31}$$

We now study the extrapolation of this $\phi(x,t)$ from $x \to \infty$ to $x \to -\infty$. Taking the $x \to \infty$ limit, we have

$$\xi = \exp\left[(ab)^{1/2}\gamma(x-vt)\right] \to \infty, \qquad \phi = \frac{4}{b}\tan^{-1}\xi = \frac{2\pi}{b} = \phi_{n=1}^{(\min)}. \tag{15.32}$$

Taking the $x \to -\infty$ limit

$$\xi = \exp\left[(ab)^{1/2}\gamma(x-vt)\right] \to 0, \qquad \phi = \frac{4}{b}\tan^{-1}\xi = 0 = \phi_{n=0}^{(\min)}. \tag{15.33}$$

Thus, the configuration

$$\phi(x,t) = \frac{4}{b}\tan^{-1}\left\{\exp\left[\pm(ab)^{1/2}\frac{(x-vt)}{\left(1-v^2\right)^{1/2}}\right]\right\} \tag{15.34}$$

interpolates between two minima of $V(\phi)$.

**(d)** The conjugate momentum is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial_0 \phi} = \partial_0 \phi \tag{15.35}$$

and the Hamiltonian density is then

$$\mathcal{H} = \pi\, \partial_0 \phi - \mathcal{L} = \tfrac{1}{2}(\partial_0 \phi)^2 + \tfrac{1}{2}(\partial_x \phi)^2 + V(\phi). \tag{15.36}$$

The total energy $E$ for the time-independent field is then

$$E = \int dx \left[\frac{1}{2}(\partial_x \phi)^2 + V(\phi)\right]. \tag{15.37}$$

Because eqn (15.9) can be written via eqn (15.29) as

$$\left(\partial_0^2 - \partial_x^2\right)\phi + \frac{\partial V}{\partial \phi} = 0, \tag{15.38}$$

for the static case $\partial_0 \phi = 0$, we get

$$-\partial_x^2 \phi + \frac{\partial V}{\partial \phi} = 0 \qquad \text{or} \qquad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial x}. \tag{15.39}$$

This implies that

$$\frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - V\right] = 0 \qquad \text{or} \qquad \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - V = c_1. \tag{15.40}$$

To calculate the constant $c_1$ we can set $x = \infty$ which gives $\xi = 0,\ \phi = 0$, hence $\partial\phi/\partial x = 0$ and $V = 0$, leading to $c_1 = 0$. This means that for the field which satisfied the equation of motion, we get

$$\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = V \qquad \text{or} \qquad \frac{dx}{d\phi} = \left(\frac{1}{2V}\right)^{1/2} \tag{15.41}$$

and the total energy is then

$$\begin{aligned} E &= \int dx \left[\frac{1}{2}(\partial_x \phi)^2 + V(\phi)\right] = \int dx 2V(\phi) = \int 2V(\phi)\frac{dx}{d\phi}d\phi \\ &= \int_0^{2\pi/b} (2V(\phi))^{1/2}\, d\phi = \left(\frac{2a}{b}\right)^{1/2} \int_0^{2\pi/b} (1 - \cos b\phi)^{1/2}\, d\phi \\ &= 8\left(\frac{a}{b^3}\right)^{1/2}. \end{aligned} \tag{15.42}$$

**(e)** From the power series $\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \cdots$ we get

$$V(\phi) = \frac{a}{b}(1 - \cos b\phi) = \frac{1}{2}ab\phi^2 - \frac{ab^3}{4!}\phi^4 + \cdots. \tag{15.43}$$

Comparing this to the standard form given by

$$V(\phi) = \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{15.44}$$

we have

$$\mu^2 = ab, \qquad \lambda = ab^3. \tag{15.45}$$

The energy is of the form

$$E = 8\left(\frac{a}{b^3}\right)^{1/2} = 8\frac{\mu^3}{\lambda}. \tag{15.46}$$

Note that in a two-dimension fields theory, the parameters $\lambda$ and $\mu^2$ have the same dimension because $\phi$ is dimensionless.

## 15.2 Planar vortex field

Consider the Higgs Lagrangian in two space and one time dimensions ($\mu = 0, 1, 2$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) + \mu^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2 \tag{15.47}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$.

**(a)** Work out the equations of motion from this Lagrangian.

**(b)** Show that the time-independent field configurations $\mathbf{A}$ and $\phi$ with $A_0 = 0$ and having the $r \to \infty$ asymptotic behaviour (in the two-dimensional polar coordinates)

$$\mathbf{A}(r,\theta) \to \frac{1}{e}\boldsymbol{\nabla}(n\theta), \quad \phi(r,\theta) \to ae^{in\theta} \qquad \text{with} \quad a = \left(\frac{\mu^2}{\lambda}\right)^{1/2} \tag{15.48}$$

will satisfy the field equations worked out in (a) upto $O(r^{-2})$.

**(c)** The magnetic flux $\Phi = \int \mathbf{B}\cdot d\mathbf{s}$ is related to the gauge field $\mathbf{A}$ of eqn (15.48) on a large circle $C$ at infinity as $\Phi = \oint_C \mathbf{A}\cdot d\mathbf{l}$. Show that this flux must appear in quantized units:

$$\Phi_n = n\frac{2\pi}{e}. \tag{15.49}$$

**Solution to Problem 15.2**

**(a)** From the Euler–Lagrange equation for the $\phi(x)$ field:

$$\frac{\partial\mathcal{L}}{\partial\phi^*} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}, \tag{15.50}$$

we obtain

$$D^\mu(D_\mu)\phi = \mu^2\phi - \lambda\phi(\phi\phi^*), \tag{15.51}$$

while the equation of motion for the $A_\mu$ field

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \tag{15.52}$$

works out to be

$$ie\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) + 2e^2 A_\mu(\phi^*\phi) = \partial^\nu F_{\mu\nu}. \tag{15.53}$$

**(b)** In the polar coordinate system, we have $\boldsymbol{\nabla} = (\hat{\mathbf{r}}\partial/\partial r,\ \partial\theta\,\hat{\boldsymbol{\theta}}/r)$. Thus for large $r$, the asymptotic form we want to use can be written as

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\theta}}\frac{ina}{r}e^{in\theta}, \qquad \mathbf{A}(r,\theta) = \frac{1}{e}\boldsymbol{\nabla}(n\theta) = \frac{n}{er}\hat{\boldsymbol{\theta}} \qquad \text{as} \quad r \to \infty. \tag{15.54}$$

Since $A_0 = 0$ and $\phi$ is independent of $t$, the full covariant derivative is given by the spatial part, which vanishes asymptotically as

$$\mathbf{D}\phi = (\boldsymbol{\nabla} - ie\mathbf{A})\phi \to \left[\hat{\boldsymbol{\theta}}\frac{ina}{r} - ie\frac{na}{er}\hat{\boldsymbol{\theta}}\right] \to O(r^{-2}) \qquad \text{as} \quad r \to \infty. \tag{15.55}$$

Also from $\phi(r,\theta) = a\,e^{in\theta}$, we have

$$\mu^2\phi - \lambda(\phi^*\phi)\phi = 0 \tag{15.56}$$

where we have used $a^2 = \mu^2/\lambda$. Thus eqn (15.51) is satisfied up to terms of order $O(r^{-2})$.

We now show that eqn (15.53) is satisfied by these field configuration. The right-hand side vanishes to $O(r^{-2})$ because the field tensor $F_{\mu\nu} \to O(r^{-2})$ as the gauge field can be written as a pure gauge, eqn (15.48)

$$A_\mu = \partial_\mu\chi \qquad \text{with} \quad \chi = \frac{n\theta}{e} \quad \text{as} \quad r \to \infty. \tag{15.57}$$

The left-hand side also vanishes to $O(r^{-2})$

$$ie\left[\frac{ina^2}{r}\hat{\boldsymbol{\theta}} - \frac{-ina^2}{r}\hat{\boldsymbol{\theta}}\right] + 2e^2\left(\frac{n}{er}\hat{\boldsymbol{\theta}}\right)a^2 = O(r^{-2}). \tag{15.58}$$

Equations (15.53) and (15.51) are satisfied because both sides of these equations vanish, at least to $O(1/r^2)$.

**(c)** From Stokes' theorem, we get

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l} \tag{15.59}$$

where $C$ is the boundary of the surface $S$. In the limit of $r \to \infty$, we have $\mathbf{A}(r,\theta) \to (1/e)\boldsymbol{\nabla}(n\theta)$, thus the components

$$A_r \to 0 \qquad \text{and} \qquad A_\theta \to \frac{1}{er} \tag{15.60}$$

and the flux quantization condition follows:

$$\Phi = \int_0^{2\pi} A_\theta r\,d\theta = \frac{n}{e}\int_0^{2\pi} d\theta = n\frac{2\pi}{e}. \tag{15.61}$$

## 15.3 Stability of soliton

The equation of motion for a scalar field in two dimensions can be written as

$$\Box\phi + \frac{\partial V}{\partial \phi} = 0 \qquad \text{where} \quad \Box = \partial_0^2 - \partial_x^2. \tag{15.62}$$

Consider a small perturbation around the time-independent solution $\phi_0(x)$,

$$\phi(x,t) = \phi_0(x) + \delta(x,t) \tag{15.63}$$

where $\delta(x,t)$ is a small quantity.

**(a)** Show that, to the first order in $\delta$, the perturbation $\delta(x,t)$ satisfies the equation

$$\Box\delta(x,t) + \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\phi_0} \delta(x,t) = 0. \tag{15.64}$$

**(b)** Take $\delta(x,t)$ in the form of a superposition of normal modes:

$$\delta(x,t) = \text{Re}\sum_n a_n\, e^{i\omega_n t}\psi_n(x). \tag{15.65}$$

Show that

$$-\frac{d^2\psi_n}{dx^2} + V'(\phi_0)\psi_n = \omega_n^2\psi_n. \tag{15.66}$$

**(c)** Show that if $\phi_0(x)$ is a monotonic function (i.e. has no nodes), then all eigenfrequencies are non-negative.

**Solution to Problem 15.3**

**(a)** The unperturbed static solution $\phi_0(x)$ satisfies the equation

$$\Box\phi_0 + V'(\phi_0) = 0 \qquad \text{or} \qquad -\partial_x^2\phi_0 + V'(\phi_0) = 0. \tag{15.67}$$

Substitute $\phi = \phi_0 + \delta$ into the field equation, $\Box(\phi_0 + \delta) + V'(\phi_0 + \delta) = 0$. For small $\delta$ we can expand $V'$

$$V'(\phi_0 + \delta) = V'(\phi_0) + V''(\phi_0)\delta. \tag{15.68}$$

Then the equation of motion is, for small $\delta$,

$$\Box\phi_0 + V'(\phi_0) + \Box\delta + V''(\phi_0)\delta = 0 \tag{15.69}$$

or

$$\Box\delta + V''(\phi_0)\delta = 0. \tag{15.70}$$

**(b)** Substituting the normal mode expansion $\delta(x,t) = \text{Re}\sum_n a_n\, e^{i\omega_n t}\,\psi_n(x)$ into the above equation, we get

$$-\frac{d^2\psi_n}{dx^2} + V'(\phi_0)\psi_n = \omega_n^2\psi_n \tag{15.71}$$

which can be viewed as a Schrödinger equation, $\psi_n$ being the eigenfunction with eigen-energy $E_n = \omega_n^2$.

**(c)** From the equation for $\phi_0$,

$$-\partial_x^2\phi_0 + V'(\phi_0) = 0, \tag{15.72}$$

we get, by differentiating with respect to $x$,

$$-\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi_0}{\partial x}\right) + V''(\phi_0)\frac{\partial\phi_0}{\partial x} = 0. \tag{15.73}$$

This can also be viewed as a Schrödinger equation with $\frac{\partial\phi_0}{\partial x}$ being an eigenfunction with zero energy $E_n = \omega_n^2 = 0$. Then if $\phi_0$ is monotonic it has no nodes. It is a well-known theorem that for a one-dimensional Schrödinger equation with arbitrary potential the eigenfunction with no nodes has the lowest energy. Since this eigenfunction has zero energy, all other eigenvalues are positive. Note that with normal mode frequencies all positive for the perturbation $\delta$, the solution $\phi_0$ is stable.

## 15.4 Monopole and angular momentum

For a charged particle moving in a monopole field, the Hamiltonian is given by

$$H = -\frac{1}{2m}\mathbf{D}^2 + V(r) \qquad \text{where} \quad \mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A} \tag{15.74}$$

where $\mathbf{A}$ is the monopole vector potential given in CL-eqn (15.25).

**(a)** Show that

$$[D_i, r_j] = \delta_{ij}, \qquad [D_i, Dj] = -ieg\varepsilon_{ijk}\frac{r_k}{r^3}. \tag{15.75}$$

**(b)** Show that the angular momentum operator $\mathbf{L}$ defined by

$$\mathbf{L} = -i\mathbf{r} \times \mathbf{D} - eg\frac{\mathbf{r}}{r} \tag{15.76}$$

will have the usual commutation relations for the angular momentum operators,

$$\begin{gathered}[L_i, D_j] = i\varepsilon_{ijk}D_k, \qquad [L_i, r_j] = i\varepsilon_{ijk}r_k, \\ [L_i, L_j] = i\varepsilon_{ijk}L_k, \qquad [L_i, H] = 0.\end{gathered} \tag{15.77}$$

**(c)** From the Heisenberg equation of motion show that

$$\frac{d\mathbf{r}}{dt} = -i\frac{\mathbf{D}}{m} \tag{15.78}$$

which implies $\mathbf{L} = m\mathbf{r} \times d\mathbf{r}/dt - eg\mathbf{r}/r$.

**(d)** Using the identity

$$\mathbf{D}\cdot\mathbf{D} = (\mathbf{D}\cdot\mathbf{r})\frac{1}{r^2}(\mathbf{D}\cdot\mathbf{r}) - (\mathbf{D}\times\mathbf{r})\frac{1}{r^2}(\mathbf{D}\times\mathbf{r}) \tag{15.79}$$

and the relation

$$\mathbf{L}\cdot\mathbf{L} = -(\mathbf{r}\times\mathbf{D})^2 + e^2g^2, \tag{15.80}$$

show that the Hamiltonian on a subspace of states for a given total angular momentum is of the form

$$H_l = -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] + \left[\frac{l(l+1) - e^2g^2}{2mr^2}\right] + V. \tag{15.81}$$

**(e)** Show that the quantum number $l$ can take only values of the form,

$$l = |eg|, |eg|+1, |eg|+2, \ldots. \tag{15.82}$$

**Solution to Problem 15.4**

**(a)** The vector potential for the monopole, according to CL-eqn (15.25), has the form

$$A_r = A_\theta = 0, \qquad A_\phi = \frac{g}{r}\frac{(1-\cos\theta)}{\sin\theta} \tag{15.83}$$

which has a string on the negative $z$-axis. Expressed in Cartesian coordinates, it has the form

$$A_x = g\frac{-y}{r(r+z)}, \qquad A_y = g\frac{x}{r(r+z)}, \qquad A_z = 0. \tag{15.84}$$

Since $\mathbf{A}$ depends only on the coordinates (not on the derivatives), it is easy to see that

$$[D_i, r_j] = [\partial_i, r_j] = \delta_{ij} \tag{15.85}$$

$$\begin{aligned}[D_i, D_j] &= [\partial_i - ieA_i, \partial_j - ieA_j] = -ie[\partial_i, A_j] - ie[A_i, \partial_j] \\ &= -ie(\partial_i A_j - \partial_j A_i) = -ie\varepsilon_{ijk}B_k = -ieg\varepsilon_{ijk}\frac{r_k}{r^3}\end{aligned} \tag{15.86}$$

where we have used the fact that the monopole field is given by $B_k = gr_k/r^3$.

**(b)** Given the definition of (15.76) $\mathbf{L} = -i\mathbf{r}\times\mathbf{D} - eg\frac{\mathbf{r}}{r}$ we now compute the commutators involving $L_i s$:

$$[L_i, D_j] = \left[-i\varepsilon_{ikl}r_kD_l, D_j\right] - eg\left[\frac{r_i}{r}, D_j\right]. \tag{15.87}$$

The first term on the right-hand side is

$$-i\varepsilon_{ikl}(r_k[D_l, D_j] + [r_k, D_j]D_l) = -i\varepsilon_{ikl}\left[r_k(-ieg)\varepsilon_{ljn}\frac{r_n}{r^3} - \delta_{jk}D_l\right]$$
$$= -eg\left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}\right) + i\varepsilon_{ijl}D_l. \quad (15.88)$$

The second term on the right-hand side in eqn (15.87) is

$$-eg\frac{1}{r}[r_i, D_j] - egr_i\left[\frac{1}{r}, D_j\right] = eg\frac{\delta_{ij}}{r} - eg\frac{r_i r_j}{r^3} \quad (15.89)$$

where we have used $[D_i, f(r)] = f'(r)\partial r/\partial r_i$. Combining these two terms, we get the result

$$[L_i, D_j] = i\varepsilon_{ijk}D_k. \quad (15.90)$$

This means that $D_i$ transforms as a vector under the rotation.

The next commutator to compute is

$$[L_i, r_j] = \left[-i\varepsilon_{ikl}r_k D_l, r_j\right] - eg\left[\frac{r_i}{r}, r_j\right]$$
$$= -i\varepsilon_{ikl}r_k[D_l, r_j] = i\varepsilon_{ijk}r_k \quad (15.91)$$

which just confirms that $r_j$ is a vector.

We now need to check the basic angular momentum commutation relation

$$[L_i, L_j] = -i\varepsilon_{jkl}[L_i, r_k D_l] - eg\left[L_i, \frac{r_j}{r}\right]. \quad (15.92)$$

The first term on the right-hand side can be calculated using the commutation relations of (15.90) and (15.91):

$$\begin{aligned}
&-i\varepsilon_{jkl}([L_i, r_k]D_l + r_k[L_i, D_l]) \\
&= -i\varepsilon_{jkl}(i\varepsilon_{ikn}r_k D_l - ir_k\varepsilon_{inl}D_n) \\
&= (\delta_{ij}\delta_{nl} - \delta_{il}\delta_{nj})r_n D_l + (-\delta_{ij}\delta_{nk} + \delta_{ik}\delta_{jn})r_k D_n \\
&= \delta_{ik}\delta_{jn}r_k D_n - \delta_{il}\delta_{nj}r_n D_l = i\varepsilon_{ijm}(-i\varepsilon_{mkn}r_k D_n)
\end{aligned} \quad (15.93)$$

where we have used the identity $\varepsilon_{abc}\varepsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}$; the second term on the right-hand side in eqn (15.92) is

$$-eg\left[L_i, \frac{r_j}{r}\right] = -eg[L_i, r_j]\frac{1}{r} = -egi\varepsilon_{ijk}\frac{r_k}{r}. \quad (15.94)$$

Combining them, we get the expected result

$$[L_i, L_j] = i\varepsilon_{ijm}\left(-i\varepsilon_{mkn}r_k D_n - \frac{r_m}{r}eg\right) = i\varepsilon_{ijm}L_m, \quad (15.95)$$

showing that $L_i$s, as defined in (15.76), are indeed the angular momentum operators.

The Hamiltonian is of the form

$$H = -\frac{1}{2m}\mathbf{D}^2 + V(r) \tag{15.96}$$

and the commutator with the angular momentum operator is

$$\begin{aligned}[L_i, H] &= \frac{-1}{2m}[L_i, \mathbf{D}^2] + [L_i, V(r)] = \frac{1}{2m}[L_i, D_j D_j] \\ &= \frac{1}{2m}(i\varepsilon_{ijk}D_k D_j + i\varepsilon_{ijk}D_j D_k) = 0. \end{aligned} \tag{15.97}$$

This means $\mathbf{L}$ is conserved by the Hamiltonian which describes the motion of particle in a monopole field. These calculations verify quantum mechanically that the monopole's contribution to angular momentum is indeed given by $-eg\mathbf{r}/r$.

**(c)** From the Heisenberg equation of motion:

$$\frac{dr_i}{dt} = i[H, r_i] = i\left(-\frac{1}{2m}\right)[\mathbf{D}^2, r_i] = -\frac{iD_i}{m}. \tag{15.98}$$

Thus we can write the angular momentum $\mathbf{L}$ of eqn (15.76) as

$$\mathbf{L} = m\mathbf{r} \times \frac{d\mathbf{r}}{dt} - eg\frac{\mathbf{r}}{r}. \tag{15.99}$$

The first term is the familiar particle angular momentum. This again confirms the interpretation of the second term as the angular momentum of the electromagnetic (monopole) field.

**(d)** In the identity

$$\mathbf{D} \cdot \mathbf{D} = (\mathbf{D} \cdot \mathbf{r})\frac{1}{r^2}(\mathbf{r} \cdot \mathbf{D}) - (\mathbf{D} \times \mathbf{r})\frac{1}{r^2}(\mathbf{r} \times \mathbf{D}) \tag{15.100}$$

we have

$$\mathbf{r} \cdot \mathbf{D} = \mathbf{r} \cdot (\boldsymbol{\nabla} - ie\mathbf{A}) = \mathbf{r} \cdot \boldsymbol{\nabla} = r\frac{\partial}{\partial r} \tag{15.101}$$

where we have used $A_r = 0$. Also

$$\mathbf{D} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{D} + 3 = r\frac{\partial}{\partial r} + 3. \tag{15.102}$$

Then the first term of the identity (15.79) is

$$(\mathbf{D} \cdot \mathbf{r})\frac{1}{r^2}(\mathbf{r} \cdot \mathbf{D}) = \left(r\frac{\partial}{\partial r} + 3\right)\frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}. \tag{15.103}$$

From $[D_i, r_j] = \delta_{ij}$, we get $\mathbf{D} \times \mathbf{r} = -\mathbf{r} \times \mathbf{D}$ and

$$[(\mathbf{r} \times \mathbf{D})_1, r^2] = \left[r_2 D_3 - r_3 D_2, r_1^2 + r_2^2 + r_3^2\right] = 2r_2 r_3 - 2r_3 r_2 = 0. \tag{15.104}$$

The second term of the identity (15.79) is then

$$-(\mathbf{D} \times \mathbf{r})\frac{1}{r^2}(\mathbf{r} \times \mathbf{D}) = \frac{1}{r^2}(\mathbf{r} \times \mathbf{D})^2. \tag{15.105}$$

Combining them, we get

$$\mathbf{D}\cdot\mathbf{D} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\mathbf{r}\times\mathbf{D}\right)^2 . \tag{15.106}$$

The last factor can be related to $\mathbf{L}^2$ because the definition (15.76) leads to

$$\mathbf{L}\cdot\mathbf{L} = -(\mathbf{r}\times\mathbf{D})^2 + e^2 g^2. \tag{15.107}$$

We then have

$$\mathbf{D}\cdot\mathbf{D} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{L^2 - e^2 g^2}{r^2}. \tag{15.108}$$

The Hamiltonian is then

$$\begin{aligned} H &= -\frac{1}{2m}\mathbf{D}^2 + V(r) \\ &= -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\mathbf{L}^2 - e^2 g^2}{r^2}\right] + V(r) \end{aligned} \tag{15.109}$$

and for states with orbital angular momentum $l$, we can replace $\mathbf{L}^2$ by $l(l+1)$ to get

$$H_l = -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{l(l+1) - e^2 g^2}{r^2}\right] + V(r). \tag{15.110}$$

**(e)** We can write the usual spherical harmonics as

$$Y_l^m(\theta, \phi) = \langle \theta, \phi | l, m\rangle \tag{15.111}$$

where $|l, m\rangle$ is the eigenstate of the angular momentum operator and $|\theta, \phi\rangle$ is the eigenvector of the particle with angular position at polar angle $\theta$ and azimuthal angle $\phi$. Under the rotation characterized by Euler angles $\alpha$, $\beta$, and $\gamma$, we have

$$e^{-iL_z\alpha}\, e^{-iL_y\beta}\, e^{-iL_z\gamma}|l, m\rangle = \sum_{m'} D^{(l)}_{m'm}(\alpha, \beta, \gamma)|l, m'\rangle \tag{15.112}$$

where $D^{(l)}_{m'm}(\alpha, \beta, \gamma) = e^{im'\alpha} d^{(l)}_{m'm}(\beta) e^{-im\gamma}$ and $d^{(l)}_{m'm}(\beta)$ can be found in books on rotation group. A particular case of the above relation is, for $\alpha = \gamma = 0$ and $\beta = -\theta$,

$$e^{iL_y\theta}|l, m\rangle = \sum_{m'} d^{(l)}_{m'm}(-\theta)|l, m'\rangle. \tag{15.113}$$

On the other hand, $|\theta, \phi\rangle$ can be obtained from $|\theta = 0\rangle$ by rotations

$$|\theta, \phi\rangle = e^{-iL_z\phi}\, e^{-iL_y\theta}|\theta = 0\rangle. \tag{15.114}$$

Thus we can write the spherical harmonics as

$$\begin{aligned} Y_l^m(\theta, \phi) &= \langle \theta, \phi | l, m\rangle = \langle \theta = 0 | e^{iL_y\theta}\, e^{iL_z\phi} | l, m\rangle \\ &= \sum_{m'} e^{-im\phi} d^{(l)}_{m'm}(-\theta)\langle \theta = 0 | l, m'\rangle. \end{aligned} \tag{15.115}$$

This means that we can construct $Y_l^m(\theta, \phi)$ from $\langle \theta = 0 | l, m'\rangle$.

The constraint on the eigenvalue $l$ can now be obtained by investigating the structure of $\langle \theta = 0 | l, m'\rangle$.

**The special case of** $eg = 0$: **L** is the same as the usual rotational operator, and it is easy to see that

$$e^{-iL_z\alpha}|\theta = 0\rangle = |\theta = 0\rangle \tag{15.116}$$

because a particle in the $z$-direction ($\theta = 0$) is invariant under rotation about the $z$-axis. From this we get

$$\langle\theta = 0|e^{iL_z\alpha}|l, m'\rangle = \langle\theta = 0|l, m'\rangle. \tag{15.117}$$

On the other hand, $e^{iL_z\alpha}|l, m'\rangle = e^{im'\phi}|l, m'\rangle$ and $(e^{im'\phi} - 1)\langle\theta = 0|l, m'\rangle = 0$. This means that

$$\langle\theta = 0|l, m'\rangle \neq 0 \qquad \text{only when} \quad m' = 0, \tag{15.118}$$

which implies that the allowed values for $l$ are 0, 1, 2, 3, . . . .

**The general case of** $eg \neq 0$:

$$L_z|\theta = 0\rangle = \left[-i(\mathbf{r} \times \mathbf{D})_z - eg\left(\frac{\mathbf{r}}{r}\right)_z\right]|\theta = 0\rangle. \tag{15.119}$$

Since $\theta = 0$ corresponds to $x = y = 0$ and $z \neq 0$, we have

$$(\mathbf{r} \times \mathbf{D})_z|\theta = 0\rangle = (xD_y - yD_x)|\theta = 0\rangle = 0 \tag{15.120}$$

$$\left(\frac{\mathbf{r}}{r}\right)_z|\theta = 0\rangle = |\theta = 0\rangle. \tag{15.121}$$

Thus we get

$$L_z|\theta = 0\rangle = -eg|\theta = 0\rangle \qquad \text{and} \qquad e^{-iL_z\alpha}|\theta = 0\rangle = e^{ieg\alpha}|\theta = 0\rangle. \tag{15.122}$$

Then

$$\langle\theta = 0|e^{iL_z\alpha}|l, m'\rangle = e^{-ieg\alpha}\langle\theta = 0|l, m'\rangle = e^{-im'\alpha}\langle\theta = 0|l, m\rangle \tag{15.123}$$

or

$$e^{-i(m'-eg)\alpha}\langle\theta = 0|l, m'\rangle = 0. \tag{15.124}$$

This implies that the matrix element $\langle\theta = 0|l, m'\rangle \neq 0$ only if $m' = eg$. Since $l \geq |m'|$, we have

$$l \geq |eg| \qquad \text{or} \qquad l = |eg|,\ |eg| + 1, \ldots. \tag{15.125}$$

# 16 Instantons

## 16.1 The saddle-point method

The transition amplitude for a particle moving in a one-dimensional space, when written as the path integral, is of the form

$$\langle x_f t | x_i 0 \rangle = \langle x_f | e^{-iHt/\hbar} | x_i \rangle = N \int [dx]\, e^{iS/\hbar} \tag{16.1}$$

where

$$H = \frac{p^2}{2m} + V(x) \tag{16.2}$$

and

$$S = \int_0^t dt' \left[ \frac{m}{2} \left( \frac{dx}{dt'} \right)^2 - V(x) \right]. \tag{16.3}$$

$N$ is the normalization constant,

**(a)** Show that in the Euclidean space $t \to -i\tau$, we can write

$$\langle x_f | e^{-H\tau/\hbar} | x_i \rangle = N \int [dx]\, e^{-S_E/\hbar}$$

where

$$S_E = \int_0^\tau d\tau \left[ \frac{m}{2} \left( \frac{dx}{d\tau} \right)^2 + V(x) \right].$$

**(b)** One can use the saddle-point method to obtain a semi-classical result. Show that in the limit $\hbar \to 0$, we have

$$\begin{aligned} N \int [dx]\, e^{-S_E/\hbar} &= N e^{-S_E(x_0)/\hbar} \left[ \det(-\partial_\tau^2 + V(x_0)) \right]^{-1/2} \\ &= N e^{-S_E(x_0)/\hbar} \prod_n \frac{1}{(\lambda_n)^{1/2}} \end{aligned} \tag{16.4}$$

where $x_0(t)$ is the classical solution, obeying the equation of motion

$$\frac{\delta S}{\delta x_0} = -m \frac{d^2 x_0}{d\tau^2} + V'(x_0) = 0 \tag{16.5}$$

and $\lambda_n$s are the eigenvalues of the second derivative operator,

$$-\frac{d^2 x_n}{d\tau^2} + V''(x_0) x_n = \lambda_n x_n. \tag{16.6}$$

**(c)** Show that the matrix element $\langle x_f|e^{-H\tau/\hbar}|x_i\rangle$ for large $\tau$ is of the form

$$\langle x_f|e^{-H\tau/\hbar}|x_i\rangle \underset{\tau\to\infty}{\to} e^{-E_0\tau/\hbar}\langle x_f|0\rangle\langle 0|x_i\rangle \tag{16.7}$$

where $|0\rangle$ is the ground state with eigenvalue $E_0$ of $H$,

$$H|n\rangle = E_n|n\rangle \qquad \text{and} \qquad E_0 < E_n \qquad n \neq 0. \tag{16.8}$$

**(d)** Show that for $x_0$ which satisfies the equation of motion (16.5), the combination

$$\frac{m}{2}\left(\frac{dx_0}{d\tau}\right)^2 - V(x_0), \tag{16.9}$$

interpreted as the 'energy', is conserved.

**(e)** Show that if $x_0(\tau)$ satisfies the equation of motion (16.5), then $(dx_0/d\tau)$ is an eigenfunction of the second derivative operator, with zero eigenvalue,

$$-\frac{d^2}{d\tau^2}\left(\frac{dx_0}{d\tau}\right) + V''(x_0)\left(\frac{dx_0}{d\tau}\right) = 0. \tag{16.10}$$

**Solution to Problem 16.1**

**(a)** Making the replacement $t = -i\tau$ in eqn (16.3), we get

$$S_E = -iS = \int_0^\tau d\tau'\left[\frac{m}{2}\left(\frac{dx}{d\tau'}\right)^2 + V(x)\right]. \tag{16.11}$$

**(b)** We are interested in calculating the transition amplitude, which is proportional to $\int[dx]\, e^{-S_E/\hbar}$. Thus we need to compute $S_E$ and $[dx]$.

In the semi-classical limit $\hbar \to 0$, we can use the saddle-point method to evaluate the integral for $S_E$. The saddle point $x_0$ of $S_E$ satisfies the first derivative equation

$$\left.\frac{\delta S_E}{\delta x}\right|_{x=x_0} = -m\frac{d^2x_0}{d\tau^2} + V'(x_0) = 0. \tag{16.12}$$

Thus, near the saddle point the leading correction to the classical action is the second derivative term. We can write $S_E$ as (see, for example, Problem 1.6)

$$\begin{aligned} S_E(x) &= S_E(x_0+\eta) \\ &= S_E(x_0) + \int_0^\tau d\tau'\eta\left[-\frac{d^2\eta}{d\tau^2} + V''(x_0)\eta\right] + \cdots . \end{aligned} \tag{16.13}$$

Write

$$x(t) = x_0(t) + \sum_n c_n x_n(t) \qquad \text{or} \qquad \eta = \sum_n c_n x_n(t) \tag{16.14}$$

with $x_0(0) = x_i$, $x_0(\tau) = x_f$, and $x_n(0) = x_n(\tau) = 0$. Also $x_n$s are chosen to be orthonormal,

$$\int_0^\tau x_n(\tau')x_m(\tau')d\tau' = \delta_{nm}. \tag{16.15}$$

Then we can take the integration measure to be

$$[dx] = [d\eta] = \prod_n \frac{1}{(2\pi\hbar)^{1/2}} dc_n \tag{16.16}$$

and

$$\begin{aligned} S_E^{(2)} &= \int_0^\tau d\tau' \eta \left[ -\frac{d^2\eta}{d\tau^2} + V''(x_0)\eta \right] \\ &= \sum_{n,m} c_n c_m \int_0^\tau d\tau' x_m \left[ -\frac{d^2 x_n}{d\tau^2} + V''(x_0) x_n \right]. \end{aligned} \tag{16.17}$$

We can choose $x_n$s to be eigenfunctions of the second derivative operator

$$-\frac{d^2 x_n}{d\tau^2} + V''(x_0)x_n = \lambda_n x_n \tag{16.18}$$

to carry out the integration,

$$S_E^{(2)} = \sum_{n,m} c_n c_m \int_0^\tau x_n(\tau')x_m(\tau')\, d\tau' \lambda_n = \sum_n c_n^2 \lambda_n \tag{16.19}$$

and

$$\begin{aligned} \int [dx]\, e^{-S_E/\hbar} &= \prod_n \int \frac{dc_n}{(2\pi\hbar)^{1/2}} \exp\left[ -\sum_n \frac{\lambda_n c_n^2}{\hbar} \right] \\ &= \prod_n \frac{1}{(\lambda_n)^{1/2}} = \frac{1}{(\det O)^{1/2}} \end{aligned} \tag{16.20}$$

with $O$ being the operator:

$$O = -\frac{d^2}{d\tau^2} + V''(x_0). \tag{16.21}$$

**(c)** From $H|n\rangle = E_n|n\rangle$, we have

$$\langle x_f|e^{-H\tau/\hbar}|x_i\rangle = \sum_n \langle x_f|e^{-H\tau/\hbar}|n\rangle\langle n|x_i\rangle = \sum_n e^{-E_n\tau/\hbar}\langle x_f|n\rangle\langle n|x_i\rangle. \tag{16.22}$$

Since $E_n > E_0$ for $n \neq 0$, the ground state will dominate the sum for $\tau$ large,

$$\langle x_f|e^{-H\tau/\hbar}|x_i\rangle \to e^{-E_0\tau/\hbar}\langle x_f|0\rangle\langle 0|x_i\rangle. \tag{16.23}$$

**(d)** By multiplying $dx_0/d\tau$ to both sides of eqn (16.5), we get,

$$\frac{d}{d\tau}\left[ \frac{m}{2}\left(\frac{dx_0}{d\tau}\right)^2 - V(x_0) \right] = 0. \tag{16.24}$$

We can thus interpret

$$E = \frac{m}{2}\left(\frac{dx_0}{d\tau}\right)^2 - V(x_0) \tag{16.25}$$

which is a constant of motion as the ‘energy’ of the particle in the Euclidean space.

**(e)** By differentiating with respect to $\tau$ both sides of the equation of motion (16.5), we get

$$m\frac{d^2}{d\tau^2}\left(\frac{dx_0}{d\tau}\right) - V''\left(\frac{dx_0}{d\tau}\right) = 0. \tag{16.26}$$

This means that $dx_0/d\tau$ is the zero mode of the second derivative operator.

## 16.2 An application of the saddle-point method

**(a)** Show that for the case of a free particle, $H_0 = p^2/2m$ the transition amplitude is given in the Euclidean space by

$$\langle x_f t_0 | x_i 0\rangle = \langle x_f | e^{-H\tau/\hbar} | x_i\rangle = \left(\frac{1}{2\pi\tau}\right)^{1/2} \exp\left[-\frac{(x_f - x_i)^2 m}{2\tau\hbar}\right]. \tag{16.27}$$

**(b)** Compare this amplitude with the formula derived in Problem 16.1(b) to show that

$$N\left[\det\left(-\frac{d^2}{d\tau^2}\right)\right]^{-1/2} = \left(\frac{m\hbar}{2\pi\tau_0}\right)^{1/2}. \tag{16.28}$$

**(c)** Use the above results to find the ground state eigenfunction $\psi_0(x)$ and energy $E_0$ for the case of a simple harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2. \tag{16.29}$$

**Solution to Problem 16.2**

**(a)**

$$\begin{aligned}
\langle x_f | e^{-H_0\tau_0/\hbar} | x_i\rangle &= \int \frac{dp}{2\pi} \langle x_f | p\rangle \langle p | e^{-p^2\tau_0/2m\hbar} | x_i\rangle \\
&= \int \frac{dp}{2\pi}\, e^{-p^2\tau_0/2m\hbar}\, e^{ip(x_f - x_i)/\hbar} \\
&= \int \frac{dp}{2\pi} \exp\left[-\frac{\tau_0}{2m\hbar}p^2 + i\frac{x_f - x_i}{\hbar}p\right] \\
&= \left(\frac{m\hbar}{2\pi\tau_0}\right)^{1/2} \exp\left[-\frac{(x_f - x_i)^2 m}{2\tau_0\hbar}\right]
\end{aligned} \tag{16.30}$$

where we have used the formula for the Gaussian integral,

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2 + bx) = \left(\frac{\pi}{a}\right)^{1/2} \exp(b^2/4a). \tag{16.31}$$

**(b)** From the result (16.4), obtained in Problem 16.1, we have

$$\langle x_f, t_0 | x_i, 0\rangle = N \exp\left[-S_E\left(x_0\right)\right]\left[\det\left(-\partial_\tau^2\right) + V''(x_0)\right]^{-1/2} \tag{16.32}$$

where $x_0(t)$ is the classical trajectory with the boundary condition

$$x_0(0) = x_i, \qquad x_0(\tau_0) = x_f. \tag{16.33}$$

In the case of free particle $V(x) = 0$, we have $d^2x_0/d\tau^2 = 0$. Thus, taking into account the boundary condition, we get

$$x_0(\tau) = x_i + (x_f - x_i)\frac{\tau}{\tau_0}. \tag{16.34}$$

The classical Euclidean action is then

$$S_E(\tau_0) = \int_0^{\tau_0} d\tau' \left[\frac{m}{2}\left(\frac{dx}{d\tau'}\right)^2\right] = \frac{m}{2}\int_0^{\tau_0} d\tau'(x_f - x_i)^2\frac{1}{\tau_0^2} = \frac{m(x_f - x_i)^2}{2\tau_0}$$

and

$$\langle x_f, t_0 | x_i, 0\rangle = N \exp\left[-\frac{m(x_f - x_i)^2}{2\tau_0\hbar}\right]\left[\det\left(-\partial_\tau^2\right)\right]^{-1/2}. \tag{16.35}$$

Compare this with Part (a), we get

$$\frac{N}{\left[\det\left(-\partial_\tau^2\right)\right]^{-1/2}} = \left(\frac{m\hbar}{2\pi\tau_0}\right)^{1/2}. \tag{16.36}$$

Eigenvalues of $-\frac{d^2}{d\tau^2}$ can be obtained as follows:

$$-\frac{d^2}{d\tau^2}x_n = \varepsilon_n x_n \quad\Rightarrow\quad x_n = A\sin(\varepsilon_n)^{1/2}\tau. \tag{16.37}$$

Then the boundary condition (16.33) requires

$$x_n(\tau_0) = 0 \quad\Rightarrow\quad \varepsilon_n = \left(\frac{n\pi}{\tau_0}\right)^2. \tag{16.38}$$

Thus we have to choose the normalization constant in such a way that

$$N\prod_n\left(\frac{\tau_0}{n\pi}\right) = \left(\frac{m\hbar}{2\pi\tau_0}\right)^{1/2}. \tag{16.39}$$

**(c)** For the case $V(x) = \frac{1}{2}m\omega^2x^2$, we get $V''(x_0) = m\omega^2$. The eigenvalues are:

$$\left(-\frac{d^2}{d\tau^2} + \omega^2\right)x_n = \varepsilon_n x_n \quad\Rightarrow\quad \varepsilon_n = \left(\frac{n\pi}{\tau_0}\right)^2 + \omega^2. \tag{16.40}$$

Thus

$$\frac{N}{\left[\det(-d^2/d\tau^2+\omega^2)\right]^{1/2}} = N\prod_n \frac{1}{(\varepsilon_n)^{1/2}}$$
$$= \left[N\prod_n\left(\frac{\tau_0}{n\pi}\right)\right]\prod_n \frac{1}{\left[1+(\omega\tau_0/n\pi)^2\right]^{1/2}}$$
$$= \left(\frac{\omega m\hbar}{2\pi}\right)^{1/2}\frac{1}{(\sinh\omega\tau_0)^{1/2}} \tag{16.41}$$

where we have used the identity

$$\prod_n^{\infty}\left(1+\frac{y^2}{n^2}\right) = \frac{\sinh\pi y}{\pi y}. \tag{16.42}$$

The classical action can be calculated as follows.

$$\frac{d^2X}{d\tau^2} = \frac{\partial V}{\partial X} = \omega^2 X \quad\Rightarrow\quad X = Ae^{\omega\tau} + Be^{-\omega\tau}. \tag{16.43}$$

Using the boundary condition we get

$$X = A\sinh\omega\tau + x_i \qquad \text{with} \quad A = \frac{(x_f - x_i)}{\sinh\omega\tau_0}. \tag{16.44}$$

The Euclidean Lagrangian is then

$$L_E = \frac{m}{2}\left(\frac{dX}{d\tau}\right)^2 + \frac{1}{2}m\omega^2X^2 = \frac{1}{2}m\omega^2A^2\cosh 2\omega\tau \tag{16.45}$$

where for simplicity we have set $x_i = 0$. The Euclidean action is then

$$S_E = \int_0^{\tau} L_E\, d\tau = \frac{m\omega A^2}{4}\sinh 2\omega\tau = \frac{m\omega x_f^2}{2}\coth\omega\tau. \tag{16.46}$$

Substituting the expressions derived in eqns (16.41) and (16.46) into the transition amplitude:

$$\langle x_f = x|e^{-H\tau/\hbar}|x_i = 0\rangle = N\left[\det\left(-\partial_\tau^2\right) + V(x_0)\right]^{-1/2} e^{-S_E(x_0)/\hbar}$$
$$= \left(\frac{m\hbar\omega}{2\pi}\right)^{1/2}\frac{1}{(\sinh\omega\tau)^{1/2}}\exp\left[-\frac{m\omega x^2}{2\hbar}\coth\omega\tau\right]. \tag{16.47}$$

As $\tau_0 \to \infty$, this amplitude has the limiting value of

$$\langle x|e^{-H\tau/\hbar}|0\rangle \to \left(\frac{m\hbar\omega}{2\pi}\right)^{1/2} e^{-\omega\tau_0/2}\, e^{-m\omega x^2/2\hbar}. \tag{16.48}$$

Compare with the formula in (16.23)

$$\langle x|e^{-H\tau/\hbar}|0\rangle \to e^{-E_0\tau/\hbar}\langle x|0\rangle\langle 0|x=0\rangle = e^{-E_0\tau/\hbar}\psi_0(x)\psi_0(0), \tag{16.49}$$

we get,

$$E_0 = \frac{\hbar\omega}{2} \quad \text{and} \quad \psi_0(x)\psi_0(0) = \left(\frac{m\hbar\omega}{2\pi}\right)^{1/2} e^{-m\omega x^2/2\hbar} \tag{16.50}$$

Set $x = 0$, we get $\psi(0) = (m\omega\hbar/2\pi)^{1/4}$ so that the ground state eigenfunction for an SHO system is

$$\psi(x) = \left(\frac{m\omega\hbar}{2\pi}\right)^{1/4} e^{-m\omega x^2/2\hbar}. \tag{16.51}$$

## 16.3 A Euclidean double-well problem

In Chapter 15 we considered the double-well potential in Minkowski space–time: here we consider its Euclidean counterpart.

$$V(x) = \lambda(x^2 - a^2)^2 \tag{16.52}$$

with minimum at $x = \pm a$. This is an example of the instanton solution (with non-trivial space and time dependence) in a field theory in one space and one time dimensions.

**(a)** Show that the solution to the equation of motion (set $m = 1$ for simplicity)

$$\frac{d^2x_1}{d\tau^2} - V'(x_1) = 0 \tag{16.53}$$

with boundary conditions $x_1(\tau) \to \pm a$ as $\tau \to \pm\infty$ has zero energy

$$E = \frac{1}{2}\left(\frac{dx_1}{d\tau}\right)^2 - V(x_1) = 0. \tag{16.54}$$

Integrate this equation to show that the solution is of the form

$$x_1 = a \tanh\frac{\omega(\tau - \tau_1)}{2} \tag{16.55}$$

with $\tau_1$ some arbitrary constant. (This solution is usually referred to as the instanton centred at $\tau_0$.) Also show that the Euclidean action for this solution is

$$S_0 = \frac{\omega^3}{12\lambda} \tag{16.56}$$

where $\omega^2 = 8\lambda a^2$.

**(b)** From the zero-energy condition (16.54), show that for large $\tau$ we have $(x_1 - a) \sim e^{-\omega\tau}$. This means that instantons are well-localized objects, having a size of the order of $(1/\omega)$.

**(c)** The zero-mode eigenfunction $x_1(\tau)$ from the translational invariance is related to the classical trajectory $\bar{x}(\tau)$ by

$$x_1 = N\frac{d\bar{x}}{d\tau}. \tag{16.57}$$

Show that the normalization constant $N$ is given by

$$N = \frac{1}{(S_0)^{1/2}} \qquad \text{with} \quad S_0 = \int d\tau \left(\frac{d\bar{x}}{d\tau}\right)^2. \tag{16.58}$$

**(d)** Show that in the path integral $[dx]$ the integration over the coefficient $c_1$ of this zero mode can be converted into an integration over the location of the centre $\tau_0$:

$$\frac{1}{(2\pi\hbar)^{1/2}}dc_1 = \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} d\tau_0. \tag{16.59}$$

**(e)** Show that the one-instanton contribution to the transition matrix element is given by

$$\langle a|e^{-H\tau}|-a\rangle|_{I=1} = N\tau\left(\frac{S_0}{2\pi\hbar}\right)^{1/2} e^{-S_0/\hbar}\left(\det{}'\left[-\partial_\tau^2 + V''(x_1)\right]\right) \tag{16.60}$$

where $\det{}'$ means that the zero eigenvalue has been taken out.

## Solution to Problem 16.3

**(a)** Given that $x_1(\tau) \to a$ as $\tau \to \infty$, we must also have $dx_1/d\tau = 0$ in that limit; otherwise, $x_1(\tau)$ will not stay at $x_1 = a$. Thus as $\tau \to \infty$, we have $x_1 = a$ and $dx_1/d\tau = 0$, which implies that both the kinetic and potential energies must vanish:

$$E = \frac{1}{2}\left(\frac{dx_1}{d\tau}\right)^2 - V(x_1) = 0 \qquad \text{at} \quad \tau \to \infty. \tag{16.61}$$

Since $E$ is independent of $\tau$, we have $E = 0$ for all values of $\tau$. Using

$$V(x_1) = \lambda(x_1^2 - a^2)^2 \tag{16.62}$$

we get from $E = 0$,

$$\left(\frac{dx_1}{d\tau}\right) = (2V(x_1))^{1/2} = -(2\lambda)^{1/2}\left(x_1^2 - a^2\right). \tag{16.63}$$

The minus sign is chosen because we are interested in the region $|x_1| < a$. Integrating this equation we have

$$\int \frac{dx_1}{(x_1^2 - a^2)} = -\int (2\lambda)^{1/2}\, d\tau \qquad \text{or} \qquad \frac{1}{2a}\ln\left|\frac{x_1 - a}{x_1 + a}\right| = -(2\lambda)^{1/2}(\tau - \tau_0). \tag{16.64}$$

Or, with $\omega^2 = 8\lambda a^2$,

$$x_1(\tau) = a \tanh \frac{\omega(\tau - \tau_0)}{2}. \tag{16.65}$$

For the zero-energy solution the classical action is

$$\begin{aligned} S_0 &= \int \left[ \frac{1}{2} \left( \frac{dx_1}{d\tau} \right)^2 + V(x_1) \right] d\tau \\ &= \int 2V(x_1) d\tau = \int 2V(x_1) \frac{d\tau}{dx_1} dx_1 = \int (2V(x_1))^{1/2}\, dx_1 \\ &= (2\lambda)^{1/2} \int_{-a}^{a} \left( -x_1^2 + a^2 \right) dx_1 = (2\lambda)^{1/2} \tfrac{4}{3} a^3. \end{aligned} \tag{16.66}$$

From $\omega^2 = 8\lambda a^2$, we get

$$S_0 = (2\lambda)^{1/2} \tfrac{4}{3} \left( \frac{\omega}{2(2\lambda)^{1/2}} \right)^3 = \frac{\omega^3}{12\lambda}. \tag{16.67}$$

Having the coupling in the denominator shows that the classical action for the instanton is intrinsically a non-perturbative contribution.

**(b)** We are interested in

$$\left( \frac{dx_1}{d\tau} \right) = [2V(x_1)]^{1/2} = -(2\lambda)^{1/2} \left( x_1^2 - a^2 \right). \tag{16.68}$$

For $\tau$ large, because of the feature of $x_1 \to a$,

$$\left( \frac{dx_1}{d\tau} \right) \simeq (2\lambda)^{1/2}\, 2a(a - x_1) \simeq \omega(a - x_1) \quad \text{or} \quad x_1 - a \simeq e^{-\omega\tau}. \tag{16.69}$$

**(c)** Substituting into the normalization condition, $\int_0^\tau [x_1(\tau')]^2\, d\tau' = 1$, the translational relation between the zero mode and the classical solutions, we get

$$N^2 \int_0^\tau \left[ \frac{d\bar{x}}{d\tau'} \right]^2 d\tau' = 1. \tag{16.70}$$

On the other hand, the classical trajectory $\bar{x}(\tau)$ has an action

$$S_0 = \int_0^\tau \left( \frac{d\bar{x}}{d\tau'} \right)^2 d\tau'. \tag{16.71}$$

Thus we get $N^2 S_0 = 1$, or

$$N = \frac{1}{(S_0)^{1/2}}. \tag{16.72}$$

**(d)** Expanding $x(\tau)$ in terms of eigenfunctions $x_n(\tau)$, having eigenvalues $E_n$

$$x(\tau) = c_1 x_1(\tau) + c_2 x_2(\tau) + \cdots \tag{16.73}$$

we get for the zero mode $E_1 = 0$

$$dx(\tau) = x_1(\tau)\, dc_1. \tag{16.74}$$

On the other hand, the change induced by a small change in the location of the centre $\tau_0$ is

$$dx = \frac{d\bar{x}}{d\tau} d\tau_0. \tag{16.75}$$

Thus we get

$$\frac{d\bar{x}}{d\tau} d\tau_0 = dx = x_1(\tau)\, dc_1 = \frac{1}{(S_0)^{1/2}} \frac{d\bar{x}}{d\tau} dc_1. \tag{16.76}$$

Cancelling $d\bar{x}/d\tau$ on both sides, we get

$$dc_1 = (S_0)^{1/2}\, d\tau_0 \qquad \text{or} \qquad \frac{1}{(2\pi\hbar)^{1/2}}\, dc_1 = \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} d\tau_0. \tag{16.77}$$

**(e)** In the usual formula

$$\langle a|e^{-H\tau/\hbar}|-a\rangle = N e^{-S_E(x_1)/\hbar} \left[\det\left(-\partial_\tau^2 + V''(x_1)\right)\right]^{-1/2} \tag{16.78}$$

we can remove the zero mode in the determinant by integrating over the location of the centre of instanton,

$$\left(\frac{S_0}{2\pi\hbar}\right)^{1/2} \int_0^\tau d\tau_0 = \tau \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} \tag{16.79}$$

Then we have

$$\langle a|e^{-H\tau/\hbar}|-a\rangle = N\tau \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} e^{-S_E(x_0)/\hbar} \left[\det{}'\left(-\partial_\tau^2 + V''\right)\right]^{-1/2}. \tag{16.80}$$

**Note on the multiple instanton solution**

Since instantons, for large $\tau$, are well-localized objects, there are also approximate solutions consisting of strings of widely separated instantons and anti-instantons, centred at $\tau_1, \ldots, \tau_n$ where

$$\tau > \tau_1 > \tau_2 > \cdots > \tau_n > 0.$$

We will now evaluate the functional integral by summing all such configurations. Since these $n$ objects are widely separated, the classical action is just $S = nS_0$, where $S_0$ is the action for one instanton. Recall that for a single-well (harmonic oscillator) potential we have,

$$N\left[\det\left(-\partial_\tau^2 + \omega^2\right)\right]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega\tau/2} \tag{16.81}$$

for large $\tau$. If it were not for the small intervals containing the instantons and anti-instantons, $V''$ would be equal to $\omega^2$ over the entire time axis and give the result

(16.81) for the single-well potential. The small intervals containing instantons and anti-instantons correct this formula and can be written as

$$N\left[\det\left(-\partial_\tau^2+\omega^2\right)\right]^{-1/2}=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}K^n \tag{16.82}$$

where the factor $K$ can be determined by demanding that this formula yields the right answer for one instanton. The zero-mode integration is again converted to integration over the centres, $\tau_1, \tau_2, \ldots, \tau_n$,

$$\int_0^\tau d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3\cdots\int_0^{\tau_{n-1}}d\tau_n=\frac{\tau^n}{n!}. \tag{16.83}$$

For transition from $-a$ to $a$, the integer $n$ is odd and from $-a$ to $-a$, it is even,

$$\begin{aligned}&\left\langle -a|e^{-H\tau/\hbar}|-a\right\rangle\\&=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}\sum_{n=even}\frac{(Ke^{-S_0/\hbar}\tau)^n}{n!}\\&=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}\frac{1}{2}\big[\exp\left(Ke^{-S_{0/}/\hbar}\tau\right)\\&\quad+\exp\left(-Ke^{-S_{0/}/\hbar}\tau\right)\big].\end{aligned} \tag{16.84}$$

Similarly,

$$\begin{aligned}&\left\langle a\left|e^{-H\tau/\hbar}\right|-a\right\rangle\\&=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}\sum_{n=even}\frac{\left(Ke^{-S_0/\hbar}\tau\right)^n}{n!}\\&=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}\frac{1}{2}\big[\exp\left(Ke^{-S_0/\hbar}\tau\right)\\&\quad-\exp\left(-Ke^{-S_{0/}/\hbar}\tau\right)\big].\end{aligned} \tag{16.85}$$

Clearly, the one-instanton contribution is

$$\left\langle a\left|e^{-H\tau/\hbar}\right|-a\right\rangle_{I=1}=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}\left(Ke^{-S_0/\hbar}\tau\right). \tag{16.86}$$

Compare this with the result in Problem 16.3(e),

$$\left\langle a\left|e^{-H\tau/\hbar}\right|-a\right\rangle=N\tau\left(\frac{S_0}{2\pi\hbar}\right)^{1/2}e^{-S_E(x_1)/\hbar}\left[\det{}'\left(-\partial_\tau^2+V''\right)\right]^{-1/2} \tag{16.87}$$

we see that

$$K\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega\tau/2}=N\left[\det{}'\left(-\partial_\tau^2+\omega^2\right)\right]^{-1/2}\left(\frac{S_0}{2\pi\hbar}\right)^{1/2} \tag{16.88}$$

or

$$K = \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} \frac{\left[\det{}'\left(-\partial_\tau^2 + V''(x)\right)\right]^{-1/2}}{\left[\det\left(-\partial_\tau^2 + \omega^2\right)\right]^{-1/2}}. \tag{16.89}$$

Note that by taking $\tau$ large in $\langle \pm a|e^{-H\tau/\hbar}|a\rangle$, we can see that the lowest two energy eigenvalues are given by

$$E_\pm = \frac{\hbar\omega}{2} \pm \hbar K e^{-S_0/\hbar}. \tag{16.90}$$

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# Index