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Core-Mantle Coupling
Part I:
Electromagnetic coupling torques

Scientific Technical Report STR08/06

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Jan M. Hagedoorn & Hans Greiner-Mai

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Introduction | 1

1.1 Introduction to this Scientific Technical Report (STR)

Since several decades it is well-known that the observed decadal variations of the length of day (LOD) and polar motion cannot be completely explained by geophysical processes in atmosphere, hydrosphere and cryosphere and by external forces. Therefore, a significant part of them must be explained by processes in the deep Earth's interior. Observed correlations between decadal variations of the rotational parameters and the geomagnetic field suggest that processes in the Earth's core are responsible for the excitation of rotational variations: one part of the angular momentum of the fluid motions in the Earth's core, generating also variations of the magnetic field, is transferred to the mantle by (different) mechanisms of core-mantle coupling.

Physically possible are the electromagnetic (EM), topographic, viscous and gravitational coupling. Which of them can contribute significantly to the Earth's rotation depends on the values of the parameters of the core-mantle transition zone like the electric conductivity of the mantle respectively the topographic height of the core-mantle boundary (CMB). The magnitude of the viscous torque is very small also for extreme assumptions of the outer-core viscosity so that the viscous coupling as a possible mechanism will be conventionally ruled out. The other types of coupling are the subject of several reports, where this STR deals with the EM coupling.

In this report, we will present the complete derivation of analytical expressions of the EM coupling torque in dependence on the parameters of the fields contributing to it. For this, we choose a special set of spherically harmonic (SH) base functions and present all major steps of the derivation. Our report will be (i) closer to a lecture note than to a scientific paper and should give all readers the possibility to follow the derivations with the related details in the appendix, and can be (ii) used as a formulary for scientists working on this special field of investigation.

1.2 Introduction to the electromagnetic core-mantle coupling

EM core-mantle coupling torques are produced by two processes: (i) temporal variations of the geomagnetic field induce electric currents, $\boldsymbol{j}$, in the conducting part of the mantle, and (ii) currents produced in the core cross the CMB and leak into the mantle. The currents in the mantle produce a Lorentz force, $\boldsymbol{j} \times \boldsymbol{B}$, on it by their interaction with the geomagnetic field, $\boldsymbol{B}$. A theoretical description of these processes was first given by Rochester (1960, 1962). In the following, different author contribute to the development of a more comprehensive theory of the EM core-mantle coupling (e.g. Roberts, 1972; Stix & Roberts, 1984; Greiner-Mai, 1987, 1993; Holme, 1998a,b, 2000; Greiner-Mai et al., 2007), where the last uses partially the formalism developed here.

In the following, we explain by what our report differs from other investigation. In appendix B.4, we show that the torque computation can be reduced to a surface integral over the CMB. This means that we must know the geomagnetic field at the CMB. Because a non-zero conductivity is a precondition for the existence of electric currents, we have to determine the geomagnetic field at the CMB from its observed values at the Earth's surface by an inverse solution of the induction equation of the mantle. Recently, Ballani et al. (2002) developed an algorithm for a rigorous inversion of the mantle induction equation to infer the poloidal geomagnetic field at the CMB from its values at the Earth's surface. In this report, we use the results of this so-called non-harmonic downward continuation (NHDC), and refer for details to the literature. The formalism for the calculation of the EM torque is adapted here to this most modern method of field continuation to the CMB and differs therein from those used in earlier investigations of EM coupling.

Another difference is that we use the orthonormal complex spherical harmonic (SH) base functions (e.g. Varshalovich et al., 1989) for the SH representations (SHR) of the fields involved in the torque computation. This enables us to apply related (existent) software, e.g. to solve numerically the coupling integrals based on Clebsch-Gordan coefficients (see e.g. section 3.3). Usually, the SHR of the geomagnetic field at the Earth's surface, the coefficients of which are our input data, are real functions and are given in Schmidt's normalization. The use of these field representations (and those derived from them) requires a transformation between both variants of SHR. Beside of basic relations and angular derivatives of the orthonormal SH needed for the torque computation, these transformations are given in appendices A and E.

Finally, the toroidal magnetic field, $\boldsymbol{B}^{\mathrm{T}}$, which generates the major part of the EM torque, must be known at the CMB. To determine this $\boldsymbol{B}^{\mathrm{T}}$ in the conducting part of the mantle, we have to solve the induction equation for the toroidal field at least as an initial-boundary value problem, for which we have inferred a boundary condition at the CMB from the poloidal geomagnetic field and the velocity field at the top of the core, respectively, shown in chapter 3. For the velocity field, $\boldsymbol{u}$, at the CMB, we use values computed according to Wardinski (2004). In our report, we imply that the SH coefficients of $\boldsymbol{u}$ are given as input data.

Basic equations | 2

In the following section, we shortly summarize the basic equations used for the derivation of the electromagnetic coupling (EM) torques between the Earth's core and mantle. The EM torque is created by the Lorentz force density, $\boldsymbol{F}$

$$\boldsymbol{F} = \boldsymbol{j} \times \boldsymbol{B}, \tag{2.1}$$

where $\boldsymbol{j}$ is the density of the electric current and $\boldsymbol{B}$ is the magnetic flux. With the definition of a torque, it follows for the EM torque $\boldsymbol{L}$,

$$\boldsymbol{L} = \int_V \boldsymbol{r} \times (\boldsymbol{j} \times \boldsymbol{B}) \,\mathrm{d}V, \tag{2.2}$$

where V is the volume of the conducting mantle. The induction equation is given by

$$\mathrm{rot}\,\boldsymbol{B} = \mu_0\,\boldsymbol{j}, \tag{2.3}$$

where μ_0 is the permeability of the vacuum. This yields the following expression for the EM torque:

$$\boldsymbol{L} = \frac{1}{\mu_0} \int_V \boldsymbol{r} \times (\mathrm{rot}\,\boldsymbol{B} \times \boldsymbol{B}) \,\mathrm{d}V. \tag{2.4}$$

2.1 Magnetic stress tensor and surface integral of the EM torque

In the next step we express the EM torque by a surface integral. For the derivation, we follow basically Rochester (1962). First, we define the magnetic stress tensor $\boldsymbol{\mathcal{M}}$, and second, we derive a relation between the integrand of eq. (2.4) and $\boldsymbol{\mathcal{M}}$. We apply afterwards the tensor divergence theorem to transform the volume into a surface integral of the core-mantle boundary, Ω_{CMB}. In this section, we use Einstein summation convention for a reduced notation. We define the magnetic stress tensor as follows:

$$\boldsymbol{\mathcal{M}} = \mathcal{M}_{jm} \boldsymbol{e}_j \otimes \boldsymbol{e}_m, \tag{2.5}$$

$$\mathcal{M}_{jm} = \frac{1}{\mu_0} \left(B_j B_m - \frac{1}{2} B_k B_k \delta_{jm} \right). \tag{2.6}$$

Here denotes $\boldsymbol{e}_i$ an unit-base vector, $\otimes$ the dyadic vector product and δ_{ij} Kronecker's symbol. For the comparison with the integrand in eq. (2.4), we need the expression

$$\mathrm{div}\,\boldsymbol{\mathcal{M}} = \frac{1}{\mu_0} \left(B_j \frac{\partial}{\partial x_j} B_k + B_k \frac{\partial}{\partial x_j} B_j - \frac{1}{2} \frac{\partial (B_n B_n)}{\partial x_k} \right) \boldsymbol{e}_k$$

which reduces with

$$\mathrm{div}\,\boldsymbol{B} = 0 \tag{2.7}$$

to

$$\mathrm{div}\,\boldsymbol{\mathcal{M}} = \frac{1}{\mu_0} \left(B_j \frac{\partial}{\partial x_j} B_k - \frac{1}{2} \frac{\partial (B_n B_n)}{\partial x_k} \right) \boldsymbol{e}_k. \tag{2.8}$$

The detailed derivation for this expression is given in appendix B.1. For the term in braces in eq. (2.4), we derived in appendix B.3 the following expression:

$$\mathrm{rot}\,\boldsymbol{B} \times \boldsymbol{B} = \left(B_j \frac{\partial}{\partial x_j} B_k - \frac{1}{2} \frac{\partial (B_n B_n)}{\partial x_k} \right) \boldsymbol{e}_k. \tag{2.9}$$

The comparison of eq. (2.8) with eq. (2.9) yields

$$\operatorname{div}\boldsymbol{\mathcal{M}} = \frac{1}{\mu_0}\left(\operatorname{rot}\boldsymbol{B}\times\boldsymbol{B}\right) = \left(\boldsymbol{j}\times\boldsymbol{B}\right). \tag{2.10}$$

A symmetric tensor like $\boldsymbol{\mathcal{M}}$ fulfills the condition (see appendix B.2)

$$\boldsymbol{r}\times\operatorname{div}\boldsymbol{\mathcal{M}} = \operatorname{div}\left(\boldsymbol{r}\times\boldsymbol{\mathcal{M}}\right). \tag{2.11}$$

With this condition, we can transform the volume integral in eq. (2.4) by the tensor divergence theorem into a surface integral:

$$\begin{aligned}\boldsymbol{L} &= \int\limits_V \boldsymbol{r}\times\operatorname{div}\boldsymbol{\mathcal{M}}\mathrm{d}V,\\ &= \int\limits_V \operatorname{div}\left(\boldsymbol{r}\times\boldsymbol{\mathcal{M}}\right)\,\mathrm{d}V,\\ &= \int\limits_\Omega \left(\boldsymbol{r}\times\boldsymbol{\mathcal{M}}\right)\cdot\boldsymbol{n}\,r^2\,\mathrm{d}\Omega,\end{aligned} \tag{2.12}$$

where the infinitesimal surface element in spherical coordinates is given by

$$\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi, \tag{2.13}$$

and Ω denotes the spherical surface integral. We can now express this surface integral by the magnetic flux $\boldsymbol{B}$ using the definition of the magnetic stress tensor by eq. (2.6). The detailed derivation is given in appendix B.4 and yields

$$\boldsymbol{L} = \frac{1}{\mu_0}\int\limits_\Omega \left[(\boldsymbol{r}\times\boldsymbol{B})(\boldsymbol{B}\cdot\boldsymbol{n}) - \frac{(B)^2}{2}(\boldsymbol{r}\times\boldsymbol{n})\right] r^2\,\mathrm{d}\Omega. \tag{2.14}$$

2.2 Cartesian components of the EM torque

In the following, we derive the Cartesian components of the EM torque $\boldsymbol{L}$, whereas we use the components of the magnetic flux, $\boldsymbol{B}$ in spherical coordinates. The Cartesian components are realized for the Earth in the geocentric coordinate system. In spherical coordinates is valid

$$\boldsymbol{r}\times\boldsymbol{B} = -rB_\varphi\boldsymbol{e}_\vartheta + rB_\vartheta\boldsymbol{e}_\varphi, \tag{2.15}$$

where

$$\boldsymbol{e}_\vartheta = \boldsymbol{e}_x\cos\vartheta\cos\varphi + \boldsymbol{e}_y\cos\vartheta\sin\varphi - \boldsymbol{e}_z\sin\vartheta, \tag{2.16}$$

$$\boldsymbol{e}_\varphi = -\boldsymbol{e}_x\sin\varphi + \boldsymbol{e}_y\cos\varphi, \tag{2.17}$$

$$\boldsymbol{r} = r\,\boldsymbol{e}_r. \tag{2.18}$$

This yields for a spherical Earth with $\boldsymbol{n} = -\boldsymbol{e}_r$ and $\boldsymbol{r}\times\boldsymbol{n} = 0$ the expression

$$\boldsymbol{L} = -\frac{1}{\mu_0}\int\limits_\Omega (\boldsymbol{r}\times\boldsymbol{B})B_r\,r^2\,\mathrm{d}\Omega, \tag{2.19}$$

$$\boldsymbol{L} = -\frac{1}{\mu_0}\int\limits_\Omega (-B_\varphi\boldsymbol{e}_\vartheta + B_\vartheta\boldsymbol{e}_\varphi)B_r\,r^3\,\mathrm{d}\Omega. \tag{2.20}$$

With the relation between the Cartesian and spherical base vectors, we can derive the Cartesian components of the EM torque. Using the expression

$$\begin{aligned}\boldsymbol{L} = -\frac{1}{\mu_0}\int\limits_\Omega B_r\Big[&-B_\varphi\big(\boldsymbol{e}_x\cos\vartheta\cos\varphi + \boldsymbol{e}_y\cos\vartheta\sin\varphi - \boldsymbol{e}_z\sin\vartheta\big)\\ &+ B_\vartheta\big(-\boldsymbol{e}_x\sin\varphi + \boldsymbol{e}_y\cos\varphi\big)\Big]\,r^3\,\mathrm{d}\Omega,\end{aligned} \tag{2.21}$$

we can derive

$$L_x = \frac{1}{\mu_0} \int\limits_\Omega B_r (B_\varphi \cos\vartheta \cos\varphi + B_\vartheta \sin\varphi)\, r^3 \,\mathrm{d}\Omega, \tag{2.22}$$

$$L_y = \frac{1}{\mu_0} \int\limits_\Omega B_r (B_\varphi \cos\vartheta \sin\varphi - B_\vartheta \cos\varphi)\, r^3 \,\mathrm{d}\Omega, \tag{2.23}$$

$$L_z = -\frac{1}{\mu_0} \int\limits_\Omega B_r B_\varphi \sin\vartheta \, r^3 \,\mathrm{d}\Omega. \tag{2.24}$$

2.3 Poloidal / Toroidal decomposition

With respect to this decomposition, we mainly follow the notation of Krause & Rädler (1980). The basic idea for the poloidal and toroidal decomposition is the representation of a divergence-free vector field, like the magnetic flux $\boldsymbol{B}$, by two scalar functions S and T. It is

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}^{\mathsf{P}} + \boldsymbol{B}^{\mathsf{T}}, \\ \boldsymbol{B} &= \operatorname{rot}\operatorname{rot}(\boldsymbol{r}S) + \operatorname{rot}(\boldsymbol{r}T), \end{aligned} \tag{2.25}$$

where the scalar functions are normalized by

$$\int\limits_\Omega S(r,\Omega)\,\mathrm{d}\Omega = 0, \tag{2.26}$$

$$\int\limits_\Omega T(r,\Omega)\,\mathrm{d}\Omega = 0. \tag{2.27}$$

In appendix B.5, we derive in eq. (B.18) and eq. (B.16)

$$\operatorname{rot}\operatorname{rot}\big(\boldsymbol{r}S\big) = -\boldsymbol{r}\Delta S + \operatorname{grad}\left(\frac{\partial}{\partial r} rS\right), \tag{2.28}$$

$$\operatorname{rot}\big(\boldsymbol{r}T\big) = -\boldsymbol{r} \times \operatorname{grad} T, \tag{2.29}$$

and with these expressions, we find for eq. (2.25)

$$\boldsymbol{B} = -\boldsymbol{r}\Delta S + \operatorname{grad}\left(\frac{\partial}{\partial r} rS\right) - \boldsymbol{r} \times \operatorname{grad} T. \tag{2.30}$$

We describe in appendix B.6 how to express eq. (2.30) in componental form in spherical coordinates and we find:

$$\begin{aligned} B_r &= -\frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\big(\sin\vartheta \frac{\partial}{\partial\vartheta} S\big) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} S\right), \\ &= -\frac{1}{r}\,\Delta_\Omega\, S, \end{aligned} \tag{2.31}$$

$$B_\vartheta = \frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{\partial}{\partial r}(rS)\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} T, \tag{2.32}$$

$$B_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}(rS)\right) - \frac{\partial}{\partial\vartheta} T. \tag{2.33}$$

In addition, we present in appendix B.7 the SHR of the poloidal and toroidal parts of the componental form of $\boldsymbol{B}$.

Starting from eqs. (2.22)–(2.24) and using the componental form of $\boldsymbol{B}$ in eqs. (2.31)–(2.33), we end up with the Cartesian components for the EM torque represented by the generating scalar functions S

and T:

$$L_x = -\frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\frac{1}{r}\left[\cot\vartheta\cos\varphi\frac{\partial^2}{\partial\varphi\partial r}(rS) + \sin\varphi\frac{\partial^2}{\partial\vartheta\partial r}(rS)\right]\right.$$
$$\left. + \left[\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}T - \cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta}T\right]\right) r^2\,\mathrm{d}\Omega, \tag{2.34}$$

$$L_y = -\frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\frac{1}{r}\left[\cot\vartheta\sin\varphi\frac{\partial^2}{\partial\varphi\partial r}(rS) - \cos\varphi\frac{\partial^2}{\partial\vartheta\partial r}(rS)\right]\right.$$
$$\left. - \left[\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}T + \cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta}T\right]\right) r^2\,\mathrm{d}\Omega, \tag{2.35}$$

$$L_z = \frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\frac{1}{r}\frac{\partial^2}{\partial\varphi\partial r}(rS) - \sin\vartheta\frac{\partial}{\partial\vartheta}T\right) r^2\,\mathrm{d}\Omega. \tag{2.36}$$

Furthermore, we can decompose the Cartesian components of the EM torque in poloidal and toroidal contributions according to the generating scalar functions S and T. The poloidal contribution to the EM torque in eqs. (2.34) – (2.36) is given by

$$L_x^{\mathsf{P}} = -\frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\cot\vartheta\cos\varphi\frac{\partial^2}{\partial\varphi\partial r}(rS) + \sin\varphi\frac{\partial^2}{\partial\vartheta\partial r}(rS)\right) r\,\mathrm{d}\Omega, \tag{2.37}$$

$$L_y^{\mathsf{P}} = -\frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\cot\vartheta\sin\varphi\frac{\partial^2}{\partial\varphi\partial r}(rS) - \cos\varphi\frac{\partial^2}{\partial\vartheta\partial r}(rS)\right) r\,\mathrm{d}\Omega, \tag{2.38}$$

$$L_z^{\mathsf{P}} = \frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\frac{\partial^2}{\partial\varphi\partial r}(rS) r\,\mathrm{d}\Omega. \tag{2.39}$$

The corresponding toroidal contribution to the EM torque in eqs. (2.34) – (2.36) is given by

$$L_x^{\mathsf{T}} = -\frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}T - \cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta}T\right) r^2\,\mathrm{d}\Omega, \tag{2.40}$$

$$L_y^{\mathsf{T}} = \frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\left(\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}T + \cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta}T\right) r^2\,\mathrm{d}\Omega, \tag{2.41}$$

$$L_z^{\mathsf{T}} = -\frac{1}{\mu_0}\int\limits_\Omega \Delta_\Omega\, S\sin\vartheta\frac{\partial}{\partial\vartheta}T r^2\,\mathrm{d}\Omega. \tag{2.42}$$

These equations are the base for the analytical solutions for the Cartesian components of the EM torque, which are derived in section 2.4.

2.4 Analytical solution for the componental form of the EM torque

2.4.1 Axial poloidal EM torque

To integrate eq. (2.39), we calculate the derivatives $\Delta_\Omega\, S$ and $\frac{\partial}{\partial\varphi}(rS)$ using the SHR of S, which is given by

$$S(r,\Omega) = \sum_{jm} S_{jm}(r)Y_{jm}(\Omega). \tag{2.43}$$

Here, $\sum\limits_{jm}$ denotes the double summation

$$\sum_{jm} = \sum_{j=1}^{j_{\max}}\sum_{m=-j}^{j}. \tag{2.44}$$

Using eq. (A.18), we find that

$$\Delta_\Omega\, S(r,\Omega) = \Delta_\Omega \sum_{jm} S_{jm}(r) Y_{jm}(\Omega) = -\sum_{jm} j(j+1) S_{jm} Y_{jm}(\Omega). \tag{2.45}$$

In addition, we derive $\frac{\partial^2}{\partial\varphi\partial r}(rS)$, where we use eq. (A.19):

$$\begin{aligned}
\frac{\partial^2}{\partial\varphi\partial r}\big(rS(r)\big) &= \frac{\partial^2}{\partial r\partial\varphi}\Big[r\sum_{jm} S_{jm}(r) Y_{jm}(\Omega)\Big],\\
&= \frac{\partial}{\partial r}\Big[r\sum_{jm} S_{jm}(r)\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\Big],\\
&= \sum_{jm} i\, m \frac{\partial}{\partial r}\big(rS_{jm}(r)\big) Y_{jm}(\Omega).
\end{aligned} \tag{2.46}$$

With these expressions, we are now able to integrate eq. (2.39) for a fixed r over the related spherical surface, Ω, applying the orthonormality of the SH:

$$L_z^{\mathsf{P}} = \frac{-1}{\mu_0}\int_\Omega \Big[\sum_{jm} j(j+1) S_{jm}(r) Y_{jm}(\Omega)\Big]\Big[\sum_{kl} i\, l \frac{\partial}{\partial r}\big(rS_{kl}(r)\big) Y_{kl}(\Omega)\Big]\, r\, \mathrm{d}\Omega\,.$$

With the substitution $\nu = -l$ and eq. (A.7) follows

$$\begin{aligned}
L_z^{\mathsf{P}} &= \frac{1}{\mu_0}\int_\Omega \sum_{jm}\sum_{k\nu} (-1)^\nu j(j+1)\, i\, \nu S_{jm}(r) \frac{\partial}{\partial r}\big(rS_{k\,-\nu}(r)\big) Y_{jm}(\Omega) Y^*_{k\nu}(\Omega)\, r\, \mathrm{d}\Omega,\\
&= \frac{r}{\mu_0}\sum_{jm}\sum_{k\nu} \delta_{jk}\delta_{m\nu} (-1)^\nu j(j+1)\, i\, \nu S_{jm}(r) \frac{\partial}{\partial r}\big(rS_{k\,-\nu}(r)\big),\\
&= \frac{r}{\mu_0}\sum_{jm} (-1)^m m\, j(j+1)\, i\, S_{jm}(r) \frac{\partial}{\partial r}\big(rS_{j\,-m}(r)\big).
\end{aligned}$$

Moreover, we can use the definition of the complex conjugate of the coefficients given in eq. (A.41), which leads to

$$L_z^{\mathsf{P}} = \frac{r}{\mu_0}\sum_{jm} m\, j(j+1)\, i\, S_{jm}(r) \frac{\partial}{\partial r}\big(rS^*_{jm}(r)\big). \tag{2.47}$$

Using the product rule to perform the partial derivative,

$$\frac{\partial}{\partial r}\big(rS^*_{jm}(r)\big) = S^*_{jm}(r) + r\frac{\partial}{\partial r} S^*_{jm}(r)\,, \tag{2.48}$$

yields

$$L_z^{\mathsf{P}} = \frac{r}{\mu_0}\sum_{jm} m\, j(j+1)\, i\, S_{jm}(r)\Big[S^*_{jm}(r) + R_{\mathsf{CMB}}\frac{\partial}{\partial r} S^*_{jm}(r)\Big]. \tag{2.49}$$

The relations between S_{jm} and the Gauss coefficients in appendix E.2 are valid only for $m > 0$, therefore, we derive the following expression using the definition of the complex conjugate of the coefficients given in eq. (A.41) and the substitution $\nu = -m$:

$$\begin{aligned}
L_z^{\mathsf{P}} = \frac{r}{\mu_0}\sum_{j=1}^{j_{\mathsf{max}}}\Bigg\{&\sum_{m=1}^{j} m\, j(j+1)\, i\, \Big[S_{jm}(r) S^*_{jm}(r) + R_{\mathsf{CMB}} S_{jm}(r)\frac{\partial}{\partial r} S^*_{jm}(r)\Big]\\
&+ \sum_{\nu=1}^{j} -\nu j(j+1)\, i\, \Big[(-1)^{2\nu} S^*_{j\nu}(r) S_{j\nu}(r) + R_{\mathsf{CMB}}(-1)^{2\nu} S^*_{j\nu}(r)\frac{\partial}{\partial r} S_{j\nu}(r)\Big]\Bigg\}.
\end{aligned}$$

This expression reduces with the substitution $\nu = m$ to

$$L_z^{\mathsf{P}} = \frac{r^2}{\mu_0} \sum_{j=1}^{j_{\max}} \sum_{m=1}^{j} m\, j(j+1)\, i \left[S_{jm}(r) \frac{\partial}{\partial r} S_{jm}^*(r) - S_{jm}^*(r) \frac{\partial}{\partial r} S_{jm}(r) \right],$$

where the SHR coefficients can be expressed by the related Gauss coefficients (see appendix E.2). For a further simplification, which also shows that $L_z^{\mathsf{P}} \in \mathbb{R}$, we need the following relation between any $a, b \in \mathbb{C}$:

$$i\,(ab^* - a^* b) = -2\ \mathrm{Im}(ab^*). \tag{2.50}$$

Applying this relation to the last expression of the axial poloidal EM torque leads to

$$L_z^{\mathsf{P}} = \frac{-2r^2}{\mu_0} \sum_{j=1}^{j_{\max}} \sum_{m=1}^{j} m\, j(j+1)\ \mathrm{Im}\Big(S_{jm}(r) \frac{\partial}{\partial r} S_{jm}^*(r) \Big). \tag{2.51}$$

We perform the computation of the coupling torque at the CMB, hence $r = R_{\mathsf{CMB}}$, for which also the Gauss coefficients are provided.

2.4.2 Axial toroidal EM torque

To integrate eq. (2.42), we have to calculate the derivatives $\Delta_\Omega\, S$ and $\sin\vartheta \frac{\partial}{\partial\vartheta} T$, where the first is calculated in section 2.4.1 and is given in eq. (2.45). The second derivative is calculated using the spherical harmonic representation of T ,

$$T(r,\Omega) = \sum_{jm} T_{jm}(r) Y_{jm}(\Omega), \tag{2.52}$$

and eq. (A.21):

$$\begin{aligned} \sin\vartheta \frac{\partial}{\partial\vartheta} T(r,\Omega) &= \sum_{jm} T_{jm}(r) \sin\vartheta \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega), \\ &= \sum_{jm} T_{jm}(r) \Bigg[j \sqrt{\frac{(j+m)^2 - m^2}{(2j+1)(2j+3)}} Y_{(j+1)\,m}(\Omega) \\ &\quad - (j+1) \sqrt{\frac{j^2 - m^2}{(2j+1)(2j-1)}} Y_{(j-1)\,m}(\Omega) \Bigg]. \end{aligned} \tag{2.53}$$

The integration over the spherical surface, Ω, with a fixed r, in eq. (2.42) can now be solved by using the orthonormality of the SH:

$$\begin{aligned} L_z^{\mathsf{T}} &= \frac{1}{\mu_0} \int_\Omega \Bigg[\sum_{jm} j(j+1) S_{jm}(r) Y_{jm}(\Omega) \Bigg] \Bigg[\sum_{kl} T_{kl}(r) \Bigg(k \sqrt{\frac{(k+l)^2 - l^2}{(2k+1)(2k+3)}} Y_{(k+1)\,l}(\Omega) \\ &\quad - (k+1) \sqrt{\frac{k^2 - l^2}{(2k+1)(2k-1)}} Y_{(k-1)\,l}(\Omega) \Bigg) \Bigg] r^2\, \mathrm{d}\Omega, \\ &= \frac{1}{\mu_0} \Bigg[\sum_{jm} \sum_{kl} j(j+1) k \sqrt{\frac{(k+l)^2 - l^2}{(2k+1)(2k+3)}} S_{jm}(r) T_{kl}(r) \int_\Omega Y_{jm}(\Omega) Y_{(k+1)\,l}(\Omega) \\ &\quad - \sum_{jm} \sum_{kl} j(j+1)(k+1) \sqrt{\frac{k^2 - l^2}{(2k+1)(2k-1)}} S_{jm}(r) T_{kl}(r) \int_\Omega Y_{jm}(\Omega) Y_{(k-1)\,l}(\Omega) \Bigg] r^2\, \mathrm{d}\Omega, \end{aligned}$$

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and with the substitution $\nu = -l$ and eq (A.7) follows

$$
\begin{aligned}
L_z^{\mathsf{T}} = \frac{r^2}{\mu_0}\Bigg[\sum_{jm}\sum_{k\nu}\Bigg(j(j+1)k\sqrt{\frac{(k+\nu)^2-\nu^2}{(2k+1)(2k+3)}}\,S_{jm}(r)\,T_{k\,-\nu}(r)\,(-1)^{\nu}\delta_{j\,(k+1)}\,\delta_{m\nu}\Bigg)\\
-\sum_{jm}\sum_{k\nu}\Bigg(j(j+1)(k+1)\sqrt{\frac{k^2-\nu^2}{(2k+1)(2k-1)}}\,S_{jm}(r)\,T_{k\,-\nu}(r)\,(-1)^{\nu}\delta_{j\,(k-1)}\,\delta_{m\nu}\Bigg)\Bigg],\\
= \frac{r^2}{\mu_0}\sum_{jm}(-1)^m\,j(j+1)\Bigg[(j-1)\sqrt{\frac{(j-m-1)^2-m^2}{(2j+1)(2j-1)}}\,S_{jm}(r)\,T_{(j-1)\,-m}(r)\\
-(j+2)\sqrt{\frac{(j+1)^2-m^2}{(2j+3)(2j+1)}}\,S_{jm}(r)\,T_{(j+1)\,-m}(r)\Bigg].
\end{aligned}
$$

Using the definition of complex conjugate coefficients in eq. (A.41) yields

$$
\begin{aligned}
L_z^{\mathsf{T}} = \frac{r^2}{\mu_0}\sum_{jm}j(j+1)\Bigg[(j-1)\sqrt{\frac{(j-m-1)^2-m^2}{(2j+1)(2j-1)}}\,S_{jm}(r)\,T^*_{(j-1)\,m}(r)\\
-(j+2)\sqrt{\frac{(j+1)^2-m^2}{(2j+3)(2j+1)}}\,S_{jm}(r)\,T^*_{(j+1)\,m}(r)\Bigg].
\end{aligned}
\tag{2.54}
$$

The relation between the Gauss coefficients and the SHR coefficients, given in eq. (E.6), is valid only for $m \geq 0$. Therefore, we derive (see appendix C.1) the following expression for L_z^{T}, where the summation is only over $m \geq 0$,

$$
\begin{aligned}
L_z^{\mathsf{T}} = \frac{r^2}{\mu_0}\sum_{j=1}^{j_{\text{max}}}j(j+1)\Bigg[\Bigg((j-1)\sqrt{\frac{(j-1)^2}{(2j+1)(2j-1)}}S_{j0}(r)T^*_{(j-1)\,0}(r)\\
-(j+2)\sqrt{\frac{(j+1)^2}{(2j+3)(2j+1)}}S_{j0}(r)T^*_{(j+1)\,0}(r)\Bigg)\\
+\sum_{m=1}^{j}(j-1)\Bigg(\sqrt{\frac{(j-m-1)^2-m^2}{(2j+1)(2j-1)}}S_{jm}(r)T^*_{(j-1)\,m}(r)\\
+\sqrt{\frac{(j+m-1)^2-m^2}{(2j+1)(2j-1)}}S^*_{jm}(r)T_{(j-1)\,m}(r)\Bigg)\\
-(j+2)\sqrt{\frac{(j+1)^2-m^2}{(2j+3)(2j+1)}}\Big(S_{jm}(r)T^*_{(j+1)\,m}(r)+S^*_{jm}(r)T_{(j+1)\,m}(r)\Big)\Bigg].
\end{aligned}
\tag{2.55}
$$

We perform the computation of the coupling torque at the CMB, hence it is $r = R_{\text{CMB}}$, for which also the Gauss coefficients are provided.

2.4.3 Non-axial poloidal EM torques

Both non-axial poloidal components of the EM torque are combined in the following complex expression

$$
\boldsymbol{L}^{\mathsf{P}} = L_x^{\mathsf{P}} + iL_y^{\mathsf{P}},
\tag{2.56}
$$

where the components are given by eqs. (2.37) and (2.38), respectively. This complex combination of the x- and y-component leads to the expression:

$$
\boldsymbol{L}^{\mathsf{P}} = -\frac{1}{\mu_0}\int_{\Omega}\Delta_\Omega\,S\Big[\cot\vartheta\frac{\partial^2}{\partial r\partial\vartheta}(rS)(\cos\varphi + i\,\sin\varphi) + \frac{\partial^2}{\partial r\partial\vartheta}(rS)(\sin\varphi - i\,\cos\varphi)\Big]r\,\mathrm{d}\Omega.
\tag{2.57}
$$

With the relations between the exponential and the trigonometric functions,

$$e^{i\varphi} = \cos\varphi + i\,\sin\varphi, \tag{2.58}$$

$$-i\,e^{i\varphi} = \sin\varphi - i\,\cos\varphi, \tag{2.59}$$

we can reduce eq. (2.57) to

$$\boldsymbol{L}^{\mathsf{P}} = -\frac{1}{\mu_0}\int\limits_{\Omega}(\Delta_\Omega\,S)e^{i\varphi}\left[\cot\vartheta\,\frac{\partial^2}{\partial r\partial\varphi}(rS) - i\,\frac{\partial^2}{\partial r\partial\vartheta}(rS)\right]r\,\mathrm{d}\Omega. \tag{2.60}$$

Moreover, we use the SHR of S, which is given by eq. (2.43), and perform the derivatives, where $\Delta_\Omega\,S$ is given by eq. (2.45). The other derivatives can be expressed by

$$\frac{\partial^2}{\partial r\partial\varphi}(rS) = \sum_{kl} i\,l\left[S_{kl}(r) + r\frac{\partial}{\partial r}S_{kl}(r)\right]Y_{kl}(\Omega), \tag{2.61}$$

$$\begin{aligned}\frac{\partial^2}{\partial r\partial\vartheta}(rS) &= \sum_{kl}\frac{1}{2}\left[S_{kl}(r) + r\frac{\partial}{\partial r}S_{kl}(r)\right]\left[\sqrt{k(k+1)-l(l+1)}e^{-i\varphi}Y_{k\,(l+1)}(\Omega)\right.\\ &\quad\left.-\sqrt{k(k+1)-l(l-1)}e^{i\varphi}Y_{k\,(l-1)}(\Omega)\right]\end{aligned} \tag{2.62}$$

Here, we have used eqs. (A.19) and (A.20) from the appendix A.1 and find with these derivatives for the complex combined torque $\boldsymbol{L}^{\mathsf{P}}$ for a fixed r:

$$\begin{aligned}\boldsymbol{L}^{\mathsf{P}} = -\frac{r}{\mu_0}\int\limits_{\Omega}&\left\{-\sum_{jm} j(j+1)S_{jm}(r)e^{i\varphi}Y_{jm}(\Omega)\right\}\left\{\left[\sum_{kl} i\,l\left(S_{kl}(r) + r\frac{\partial}{\partial r}S_{kl}(r)\right)\right.\right.\\ &\left.\cdot\cot\vartheta Y_{kl}(\Omega)\right] - \left[\frac{i}{2l}\sum_{kl}\left(S_{kl}(r) + R_{\mathsf{CMB}}\frac{\partial}{\partial r}S_{kl}(r)\right)\left(\sqrt{k(k+1)-l(l+1)}e^{-i\varphi}Y_{k\,(l+1)}(\Omega)\right.\right.\\ &\left.\left.\left.-\sqrt{k(k+1)-l(l-1)}e^{i\varphi}Y_{k\,(l-1)}(\Omega)\right)\right]\right\}\mathrm{d}\Omega.\end{aligned} \tag{2.63}$$

We introduce a few abbreviations for the further derivation:

$$\mathcal{S}_{kl} = \left(S_{kl}(r) + r\frac{\partial}{\partial r}S_{kl}(r)\right), \tag{2.64}$$

$$\mathcal{W}^{+}_{kl} = \sqrt{k(k+1)-l(l+1)}, \tag{2.65}$$

$$\mathcal{W}^{-}_{kl} = \sqrt{k(k+1)-l(l-1)}. \tag{2.66}$$

In addition, we take into account (Varshalovich et al., 1989, Sec. 5.7),

$$\cot\vartheta Y_{kl}(\Omega) = -\frac{1}{2l}\left[\sqrt{k(k+1)-l(l+1)}e^{-i\varphi}Y_{k\,(l+1)}(\Omega) + \sqrt{k(k+1)-l(l-1)}e^{i\varphi}Y_{k\,(l-1)}(\Omega)\right], \tag{2.67}$$

to derive the expression,

$$\begin{aligned}\boldsymbol{L}^{\mathsf{P}} = -\frac{r}{\mu_0}\int\limits_{\Omega}&\left\{\sum_{jm} j(j+1)S_{jm}(r)e^{i\varphi}Y_{jm}(\Omega)\right\}\left\{\frac{i}{2}\sum_{kl}\mathcal{S}_{kl}\left[\mathcal{W}^{+}_{kl}e^{-i\varphi}Y_{k\,(l+1)}(\Omega)\right.\right.\\ &\left.\left.+\,\mathcal{W}^{-}_{kl}e^{i\varphi}Y_{k\,(l-1)}(\Omega)\right] + \frac{i}{2}\sum_{kl}\mathcal{S}_{kl}\left[\mathcal{W}^{+}_{kl}e^{-i\varphi}Y_{k\,(l+1)}(\Omega) - \mathcal{W}^{-}_{kl}e^{i\varphi}Y_{k\,(l-1)}(\Omega)\right]\right\}\mathrm{d}\Omega,\end{aligned}$$

which reduces to

$$\boldsymbol{L}^{\mathsf{P}} = -\frac{i\,r}{\mu_0}\int\limits_{\Omega}\left\{\sum_{jm} j(j+1)\,S_{jm}(r)\,e^{i\varphi}\,Y_{jm}(\Omega)\right\}\left\{\sum_{kl}\mathcal{S}_{kl}\,\mathcal{W}^{+}_{kl}\,e^{-i\varphi}\,Y_{k\,(l+1)}(\Omega)\right\}\mathrm{d}\Omega. \tag{2.68}$$

Now we use one symbol for the summation over all four indices and apply the definition of the complex conjugate in eq. (A.7) to $Y_{k\,(l+1)}$, obtaining

$$\begin{aligned}\boldsymbol{L}^{\mathsf{P}} &= -\frac{i\,r}{\mu_0}\int\limits_{\Omega}\sum_{jmkl} j(j+1)\,\mathcal{W}^{+}_{kl}\,S_{jm}(r)\,\mathcal{S}_{kl}\,Y_{jm}(\Omega)(-1)^{(l+1)}Y^{*}_{k\,-(l+1)}(\Omega)\,\mathrm{d}\Omega,\\ &= -\frac{i\,r}{\mu_0}\sum_{jmkl}(-1)^{(l+1)}j(j+1)\,\mathcal{W}^{+}_{kl}\,S_{jm}(r)\,\mathcal{S}_{kl}\int\limits_{\Omega}Y_{jm}(\Omega)\,Y^{*}_{k\,-(l+1)}(\Omega)\,\mathrm{d}\Omega,\end{aligned} \tag{2.69}$$

where the integral over Ω can be solved using the orthonormality relation given by eq. (A.6). The expression for $\boldsymbol{L}^{\mathsf{P}}$ reduces then to the following summation

$$\boldsymbol{L}^{\mathsf{P}} = -\frac{i\,r}{\mu_0}\sum_{jm}(-1)^{(l+1)}j(j+1)\,\mathcal{W}^{+}_{j\,-(m+1)}\,S_{jm}(r)\,\mathcal{S}_{j\,-(m+1)},$$

which can be further reduced changing the sign of the first factor by the definition of the complex conjugate SH in eq. (A.7):

$$\boldsymbol{L}^{\mathsf{P}} = \frac{i\,r}{\mu_0}\sum_{jm} j(j+1)\,\mathcal{W}^{+}_{j\,-(m+1)}\,S_{jm}(r)\,\mathcal{S}^{*}_{j\,(m+1)}.$$

In the next step, we re-substitute the used abbreviations and find:

$$\begin{aligned}\boldsymbol{L}^{\mathsf{P}} &= \frac{i\,r}{\mu_0}\sum_{jm} j(j+1)\sqrt{k(k+1)-l(l+1)}\,S_{jm}(r)\\ &\quad\cdot\left[S^{*}_{j\,(m+1)}(r)+r\frac{\partial}{\partial r}S^{*}_{j\,(m+1)}(r)\right].\end{aligned} \tag{2.70}$$

The relation between the Gauss coefficients and the SHR coefficients, given in eq. (E.6), is valid only for $m \geq 0$. Therefore, we derive (see appendix C.2) the following expression for $\boldsymbol{L}^{\mathsf{P}}$, where the summation is only over $m > 0$,

$$\begin{aligned}\boldsymbol{L}^{\mathsf{P}} &= \frac{i\,r}{\mu_0}\sum_{j=1}^{j_{\mathsf{max}}} j(j+1)\Bigg\{\sqrt{j(j+1)}S_{j0}(r)\left(S^{*}_{j1}(r)+r\frac{\partial}{\partial r}S^{*}_{j1}(r)\right)\\ &\quad+\sum_{m=1}^{j}\Bigg[\sqrt{j(j+1)-m(m+1)}S_{jm}(r)\left(S^{*}_{j\,(m+1)}(r)+R_{\mathsf{CMB}}\frac{\partial}{\partial r}S^{*}_{j\,(m+1)}(r)\right)\\ &\quad-\sqrt{j(j+1)-m(m-1)}S^{*}_{jm}(r)\left(S_{j\,(m+1)}(r)+r\frac{\partial}{\partial r}S_{j\,(m+1)}(r)\right)\Bigg]\Bigg\},\end{aligned} \tag{2.71}$$

where j_{max} denotes the maximal degree of the SHR.

2.4.4 Non-axial toroidal EM torques

In analogy to the derivation of $\boldsymbol{L}^{\mathsf{P}}$ in section 2.4.3, we define the complex combined non-axial toroidal EM torque by

$$\boldsymbol{L}^{\mathsf{T}} = L^{\mathsf{T}}_{x} + iL^{\mathsf{T}}_{y}. \tag{2.72}$$

The x- and y-components are defined in eqs. (2.40) and (2.41), respectively. With these equations, the expression for $\boldsymbol{L}^{\mathsf{T}}$ reads

$$\boldsymbol{L}^{\mathsf{T}} = \frac{1}{\mu_0}\int\limits_{\Omega}\Delta_{\Omega}\,S\left[\frac{-1}{\sin\vartheta}(\sin\varphi - i\,\cos\varphi)\frac{\partial}{\partial\varphi}T+\cos\vartheta(\cos\varphi + i\,\sin\varphi)\frac{\partial}{\partial\vartheta}T\right]r^2\,\mathrm{d}\Omega. \tag{2.73}$$

In the next step, we apply the eqs. (2.58) and (2.59) and find

$$\boldsymbol{L}^{\mathsf{T}} = \frac{1}{\mu_0}\int\limits_{\Omega}(\Delta_{\Omega}\,S)e^{i\varphi}\left[\frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi}T+\cos\vartheta\frac{\partial}{\partial\vartheta}T\right]r^2\,\mathrm{d}\Omega. \tag{2.74}$$

For the further derivation, we express S, T and their derivatives by SH. The related SHR are given in eqs. (2.45), (2.52) and (A.19). Moreover, we represent $\boldsymbol{L}^{\mathsf{T}}$ at $r = R_{\mathsf{CMB}}$, which leads to

$$\boldsymbol{L}^{\mathsf{T}} = \frac{R_{\mathsf{CMB}}^2}{\mu_0} \int\limits_{\Omega_{\mathsf{CMB}}} \left[\sum_{jm} -j(j+1) S_{jm}(r) Y_{jm}(\Omega) e^{i\varphi}\right] \left[\sum_{kl} \left(\frac{-l}{\sin\vartheta} Y_{kl}(\Omega) + \cos\vartheta \frac{\partial}{\partial\vartheta} Y_{kl}(\Omega)\right) T_{kl}(r)\right] \mathrm{d}\Omega,$$

and by using eq. (A.7) the expression above reduces to

$$\boldsymbol{L}^{\mathsf{T}} = \frac{r^2}{\mu_0} \int\limits_{\Omega} \sum_{jmkl} j(j+1) S_{jm}(r)\, T_{kl}(r) Y_{jm}(\Omega) e^{i\varphi} (-1)^l \left[\frac{l}{\sin\vartheta} Y^*_{k\,-l}(\Omega) - \cos\vartheta \frac{\partial}{\partial\vartheta} Y^*_{k\,-l}(\Omega)\right] \mathrm{d}\Omega.$$

In the next step, we split the integration over the two angular dependencies and represent the SH by their definition, given in eq. (A.1). This leads to

$$\begin{aligned}\boldsymbol{L}^{\mathsf{T}} = \frac{R_{\mathsf{CMB}}^2}{\mu_0} \sum_{jmkl} j(j+1) S_{jm}(r)\, T_{kl}(r) (-1)^l \int\limits_0^{2\pi} e^{i(m+l+1)\varphi} \int\limits_0^{\pi} P_{jm}(\cos\vartheta) \Big[l P_{k\,-l}(\cos\vartheta) \\ - \cos\vartheta \sin\vartheta \frac{\partial}{\partial\vartheta} P_{k\,-l}(\cos\vartheta)\Big] \mathrm{d}\varphi\, \mathrm{d}\vartheta. \qquad (2.75)\end{aligned}$$

Applying eq. (C.6) leads to

$$\begin{aligned}\boldsymbol{L}^{\mathsf{T}} = \frac{r^2}{\mu_0} \sum_{jmkl} j(j+1) S_{jm}(r)\, T_{kl}(r) (-1)^l \delta_{m\,-(l+1)} \int\limits_0^{\pi} P_{jm}(\cos\vartheta) \Big[l P_{k\,-l}(\cos\vartheta) \\ - \cos\vartheta \sin\vartheta \frac{\partial}{\partial\vartheta} P_{k\,-l}(\cos\vartheta)\Big] \mathrm{d}\vartheta,\end{aligned}$$

which can further be reduced by performing Kronecker's symbol and using the complex conjugate of T:

$$\begin{aligned}\boldsymbol{L}^{\mathsf{T}} = -\frac{r^2}{\mu_0} \sum_{jmk} j(j+1) S_{jm}(r)\, T^*_{k\,(m+1)}(r) \int\limits_0^{\pi} P_{jm}(\cos\vartheta) \Big[(m+1) P_{k\,(m+1)}(\cos\vartheta) \\ + \cos\vartheta \sin\vartheta \frac{\partial}{\partial\vartheta} P_{k\,(m+1)}(\cos\vartheta)\Big] d\vartheta. \qquad (2.76)\end{aligned}$$

For the integration over ϑ, we use the relation

$$\begin{aligned}K &= (m+1) P_{k\,(m+1)}(\cos\vartheta) + \cos\vartheta \sin\vartheta \frac{\partial}{\partial\vartheta} P_{k\,(m+1)}(\cos\vartheta) \\ &= \frac{\sin\vartheta}{(2k+1)} \Big[(k-m+1)(k-m) k P_{(k+1)\,m}(\cos\vartheta) + (k+m+1)(k+m)(k+1) P_{(k-1)\,m} \Big], \qquad (2.77)\end{aligned}$$

which is derived in appendix C.3 and leads to

$$\begin{aligned}\boldsymbol{L}^{\mathsf{T}} = -\frac{r^2}{\mu_0} \sum_{jmk} j(j+1) S_{jm}(r)\, T^*_{k\,(m+1)}(r) \int\limits_0^{\pi} \Big[\frac{(k-m+1)(k-m)k}{(2k+1)} P_{jm}(\cos\vartheta) P_{(k+1)\,m}(\cos\vartheta) \\ + \frac{(k+m+1)(k+m)(k+1)}{(2k+1)} P_{jm}(\cos\vartheta) P_{(k-1)\,m}(\cos\vartheta)\Big] \sin\vartheta\, \mathrm{d}\vartheta.\end{aligned}$$

With the orthogonality relation in eq. (A.38), we find

$$\begin{aligned}\boldsymbol{L}^{\mathsf{T}} = -\frac{r^2}{\mu_0} \sum_{jmk} j(j+1) S_{jm}(r)\, T^*_{k\,(m+1)}(r) \Big[\frac{(k-m+1)(k-m)k}{(2k+1)}\, \delta_{j\,(k+1)} \\ + \frac{(k+m+1)(k+m)(k+1)}{(2k+1)}\, \delta_{j\,(k-1)}\Big],\end{aligned}$$

which reduces by applying Kronecker's symbol to

$$\boldsymbol{L}^{\mathsf{T}} = -\frac{r^2}{\mu_0}\sum_{jm} j(j+1)S_{jm}(r)\left[\frac{(j-m)(j-m-1)(j-1)}{(2j-1)}\,T^*_{(j-1)\,(m+1)}(r)\right.$$
$$\left. + \frac{(j+m+2)(j+m+1)(j+2)}{(2j+3)}T^*_{(j+1)\,(m+1)}(r)\right]. \tag{2.78}$$

Due to the restriction of the relations between the Gauss coefficients and the SHR coefficients to indices $m \geq 0$, given in eq. (E.6), we have to reformulate eq. (2.78) for non-negative m. Therefore, we derive (see appendix C.3) an expression for $\boldsymbol{L}^{\mathsf{T}}$, where the summation is only over $m > 0$,

$$\boldsymbol{L}^{\mathsf{T}} = -\frac{r^2}{\mu_0}\sum_{j=1}^{j_{\text{max}}} j(j+1)\Bigg\{S_{j0}(r)\left[\frac{j(j-1)^2}{(2j-1)}T^*_{(j-1)\,1}(r) + \frac{(j+1)(j+2)^2}{(2j+3)}T^*_{(j+1)\,1}(r)\right]$$
$$+ \sum_{m=1}^{j}\Bigg[S_{jm}(r)\Bigg(\frac{(j-m)(j-m-1)(j-1)}{((2j-1))}\,T^*_{(j-1)\,(m+1)}(r)$$
$$+ \frac{(j+m+2)(j+m+1)(j+2)}{(2j+3)}\,T^*_{(j+1)\,(m+1)}(r)\Bigg)$$
$$- S^*_{jm}(r)\Bigg(\frac{(j+m)(j+m-1)(j-1)}{(2j-1)}\,T_{(j-1)\,(m-1)}(r)$$
$$+ \frac{(j-m+2)(j-m+1)(j+2)}{(2j+3)}\,T_{(j+1)\,(m-1)}(r)\Bigg)\Bigg]\Bigg\}. \tag{2.79}$$

Here, j_{max} denotes the maximal degree of the SHR of the available geomagnetic field representation. The coupling torque is computed at the CMB, hence it is $r = R_{\text{CMB}}$.

Determination of the toroidal magnetic field | 3

The relation between the Gauss coefficients and the SHR of the field generating scalar S is given in appendix E.2. In the sections above, we have expressed the different components of the EM coupling torque with series of SH coefficients of the field-generating scalars S and T. The relation between the so-called Gauss coefficients and S_{jm} allows us to determine the poloidal components of the torque. Moreover, we need to express the SH coefficients of the field-generating scalar of the toroidal magnetic field, T, to compute the toroidal components of the coupling torque. This is not possible by the magnetic field alone (and its downward continuation to the CMB), because the toroidal part of the magnetic field vanish outside of a conductor, as we will show by the boundary conditions in the following sections.

For the determination of the toroidal magnetic field at the CMB, we have to formulate the boundary value problem (BVP) for the field generating scalar, T, of the toroidal magnetic field. The related field equation is the induction equation, which has to be solved for special boundary conditions. In the next section, we present the induction equation, followed by the formulation of the BVP and the related boundary conditions for T.

3.1 Induction equation

The derivation of the induction equation in the mantle and the fluid outer core is based on the so-called Maxwell equations (e. g. Krause & Rädler, 1980, Chap. 2),

$$\mathrm{rot}\,\boldsymbol{B} = \mu_0 \boldsymbol{j}, \tag{3.1}$$

$$\mathrm{div}\,\boldsymbol{B} = 0, \tag{3.2}$$

$$\mathrm{rot}\,\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}, \tag{3.3}$$

and the Ohm's law

$$\boldsymbol{j} = \sigma \boldsymbol{E}. \tag{3.4}$$

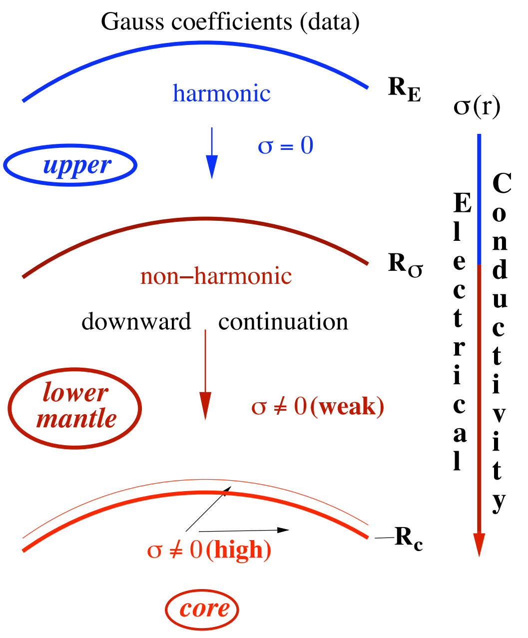

Figure 3.1: Schematic sketch of the structure of the Earth's interior related to the electric conductivity.

For the further derivation, we have to consider the different domains, distinguished by the flow of the fluid outer-core relative to the mantle (see fig. 3.1). The geomagnetic field is described in a mantle-fixed coordinate system. Hence, we have to consider in Ohm's law for the core additional contribution to the current density. The major additional contribution is created by the interaction of the geomagnetic field, described in the mantle coordinate system, and the flow of the electrically conducting fluid of the outer core. With the fluid-flow velocity denoted by $\boldsymbol{u}$, the contribution related to the large-scale flow is given by $\boldsymbol{u} \times \boldsymbol{B}$. For general description of this problem, we introduce in addition the contribution $\boldsymbol{E}^e$ related to the turbulent flow in the outer core. Hence, we have to derive separately the induction equations for the mantle and the fluid outer core. We follow here mainly the approach presented by Greiner-Mai (1986).

The current density in the conducting part of the mantle following eq. (3.4) is given by

$$\boldsymbol{j}_{\mathrm{M}} = \sigma_{\mathrm{M}} \boldsymbol{E}_{\mathrm{M}}, \tag{3.5}$$

where $\boldsymbol{E}_{\mathrm{M}}$ denotes the electric field in the mantle. We can eliminate $\boldsymbol{E}_{\mathrm{M}}$ in eq. (3.3) using the expression above, which leads to

$$\operatorname{rot} \frac{1}{\sigma_{\mathrm{M}}} \boldsymbol{j}_{\mathrm{M}} = -\frac{\partial}{\partial t} \boldsymbol{B}.$$

Next, we can represent the current density by eq. (3.1), which results in

$$\frac{1}{\mu_0} \operatorname{rot} \left(\frac{1}{\sigma_{\mathrm{M}}} \operatorname{rot} \boldsymbol{B} \right) = -\frac{\partial}{\partial t} \boldsymbol{B}. \tag{3.6}$$

Together with eq. (3.2), the set of induction equations for a conducting mantle is given for $r \in [R_{\mathrm{CMB}}, R_\sigma]$ by:

$$\begin{aligned} \frac{1}{\mu_0} \operatorname{rot} \left(\frac{1}{\sigma_{\mathrm{M}}} \operatorname{rot} \boldsymbol{B} \right) + \frac{\partial}{\partial t} \boldsymbol{B} = 0, \\ \operatorname{div} \boldsymbol{B} = 0, \end{aligned}$$

where the radius of the CMB is denoted by R_{CMB} and the upper bound of the conducting part of the mantle is denoted by R_σ .

For the fluid outer core, the expression for the current density contains the above mentioned additional contributions beside $\boldsymbol{E}_{\mathrm{C}}$, the electric field in the outer core:

$$\boldsymbol{j}_{\mathrm{C}} = \sigma_{\mathrm{C}} \big(\boldsymbol{E}_{\mathrm{C}} + \boldsymbol{u} \times \boldsymbol{B} + \boldsymbol{E}^{\mathrm{e}} \big). \tag{3.7}$$

In contrast to the mantle, we have to consider here also the velocity, $\boldsymbol{u}$, of the conducting fluid in the outer core moving relatively to the mantle. For a general description, we have to consider additionally $\boldsymbol{E}^{\mathrm{e}}$, which is associated with the turbulent flow in the conducting outer core. Analogous to the derivation for the mantle, we can represent $\boldsymbol{E}_{\mathrm{C}}$ using eqs. (3.4) and (3.1) to reformulate eq. (3.3), obtaining

$$\begin{aligned} \operatorname{rot} \left(\frac{1}{\sigma_{\mathrm{C}}} \boldsymbol{j}_{\mathrm{C}} - \big(\boldsymbol{u} \times \boldsymbol{B} + \boldsymbol{E}^{\mathrm{e}} \big) \right) &= -\frac{\partial}{\partial t} \boldsymbol{B}, \\ \frac{1}{\mu_0} \operatorname{rot} \left(\frac{1}{\sigma_{\mathrm{C}}} \operatorname{rot} \boldsymbol{B} \right) - \operatorname{rot} \big(\boldsymbol{u} \times \boldsymbol{B} + \boldsymbol{E}^{\mathrm{e}} \big) &= -\frac{\partial}{\partial t} \boldsymbol{B}. \end{aligned} \tag{3.8}$$

Together with eq. (3.2), the set of induction equations for the outer core is given for $r \in [R_{\mathrm{ICB}}, R_{\mathrm{CMB}}]$ by:

$$\begin{aligned} \frac{1}{\mu_0} \operatorname{rot} \left(\frac{1}{\sigma_{\mathrm{C}}} \operatorname{rot} \boldsymbol{B} \right) - \operatorname{rot} \big(\boldsymbol{u} \times \boldsymbol{B} + \boldsymbol{E}^{\mathrm{e}} \big) &= -\frac{\partial}{\partial t} \boldsymbol{B}, \\ \operatorname{div} \boldsymbol{B} &= 0. \end{aligned}$$

Here, R_{ICB} denotes the radius of the spherical boundary between the liquid outer core and the solid inner core.

For the further derivation, we need the induction equations for the mantle and core in their scalar form (e. g. Krause & Rädler, 1980, Sec. 14.2). For this, we use the poloidal and toroidal decomposition in eq. (2.25). Applying this to eq. (3.6) leads to the following scalar induction equations for the conducting mantle domain ($r \in [R_{\mathrm{CMB}}, R_\sigma]$):

$$\frac{1}{\mu_0 \sigma_{\mathrm{M}}} \left(\Delta T - \frac{1}{r \sigma_{\mathrm{M}}} \frac{\mathrm{d}}{\mathrm{d}r} \sigma_{\mathrm{M}} \frac{\partial}{\partial r} (rT) \right) = \frac{\partial}{\partial t} T, \tag{3.9}$$

$$\frac{1}{\mu_0 \sigma_{\mathrm{M}}} \Delta S = \frac{\partial}{\partial t} S. \tag{3.10}$$

The detailed derivation is given in appendix D.1.

For the fluid outer core, $r \in [R_{\text{ICB}}, R_{\text{CMB}}]$, we use for the derivation of the scalar induction equation (see appendix D.1) the following definitions for field-generating scalars U, V, W and U^{e}, V^{e}, W^{e},

$$(\boldsymbol{u} \times \boldsymbol{B})^{\text{T}} = \operatorname{rot}(\boldsymbol{r}U), \tag{3.11}$$

$$(\boldsymbol{u} \times \boldsymbol{B})^{\text{P}} = \boldsymbol{r}V + \operatorname{grad} W, \tag{3.12}$$

$$\boldsymbol{E}^{\text{eT}} = \operatorname{rot}(\boldsymbol{r}U^{\text{e}}), \tag{3.13}$$

$$\boldsymbol{E}^{\text{eP}} = \boldsymbol{r}V^{\text{e}} + \operatorname{grad} W^{\text{e}}, \tag{3.14}$$

and we require the normalizations

$$\int_\Omega U \, \mathrm{d}\Omega = \int_\Omega U^{\text{e}} \mathrm{d}\Omega = \int_\Omega V \, \mathrm{d}\Omega = \int_\Omega V^{\text{e}} \mathrm{d}\Omega = 0 \, . \tag{3.15}$$

For the poloidal parts in eqs. (3.12) and (3.14) more general decompositions are needed, because $\boldsymbol{u} \times \boldsymbol{B}$ is not anymore divergence free. The scalar induction equations for the outer core read

$$\frac{1}{\mu_0 \sigma_{\text{C}}} \left(\Delta T - \frac{1}{r\sigma_{\text{C}}} \frac{\mathrm{d}}{\mathrm{d}r} \sigma_{\text{C}} \frac{\partial}{\partial r} (rT) \right) + V + V^{\text{e}} = \frac{\partial}{\partial t} T, \tag{3.16}$$

$$\frac{1}{\mu_0 \sigma_{\text{M}}} \Delta S + U + U^{\text{e}} = \frac{\partial}{\partial t} S. \tag{3.17}$$

The poloidal induction equation for the mantle, eq. (3.10), is used to solve the problem of non-harmonic downward continuation of the so-called Gauss coefficients. Moreover, the toroidal induction equation for the conducting mantle domain, eq. (3.9), is the differential equation for the boundary value problem (BVP), which is set up to determine the toroidal magnetic field at the CMB. In the general case, we need for the derivation of the necessary boundary values also the toroidal induction equation (3.16). For the further derivation, all mentioned equations have to be represented by spherical harmonics. The related transformation for the spherical harmonic representation (SHR) is summarized in appendix D.2. For the toroidal induction equation of the mantle we find:

$$\begin{aligned} &\frac{\partial^2}{\partial r^2} T_{jm}(r,t) + \left[\frac{2}{r} - \frac{1}{\sigma_{\text{M}}(r)} \frac{\mathrm{d}}{\mathrm{d}r} \sigma_{\text{M}}(r) \right] \frac{\partial}{\partial r} T_{jm}(r,t) \\ &\quad - \left[\frac{j(j+1)}{r^2} + \frac{1}{r\sigma_{\text{M}}(r)} \frac{\mathrm{d}}{\mathrm{d}r} \sigma_{\text{M}}(r) \right] T_{jm}(r,t) = \mu_0 \sigma_{\text{M}}(r) \frac{\partial}{\partial t} T_{jm}(r,t). \end{aligned} \tag{3.18}$$

Analogously, we find for the corresponding poloidal induction equation:

$$\frac{\partial^2}{\partial r^2} S_{jm}(r,t) + \frac{2}{r} \frac{\partial}{\partial r} S_{jm}(r,t) - \frac{j(j+1)}{r^2} S_{jm}(r,t) = \mu_0 \sigma_{\text{M}}(r) \frac{\partial}{\partial t} S_{jm}(r,t). \tag{3.19}$$

3.2 Boundary value problem for the toroidal magnetic field

In the following two sections, we derive the boundary conditions for the field-generating scalars S and T, starting from the general continuity condition for the magnetic flux $\boldsymbol{B}$, which is valid for the time scales considered in geophysics. We summarize the boundary value problem (BVP) for this field-generating scalars in section 3.2.3.

3.2.1 Boundary conditions at the surface $r = R_\sigma$

We model the surface between the isolating part of the mantle and its conducting part by a spherical boundary at $r = R_\sigma$ (see fig. 3.1). For this boundary, the continuity of the magnetic field, $\boldsymbol{B}$, is valid, from which follows the continuity of its orthogonal poloidal and toroidal parts:

$$[\boldsymbol{B}^{\text{P}}]_-^+ = 0 \qquad \text{and} \qquad [\boldsymbol{B}^{\text{T}}]_-^+ = 0. \tag{3.20}$$

The notation $[\ldots]_{-}^{+}$ is the short form for the difference between the related values infinitesimal above $(+)$ and below $(-)$ of the considered boundary. For the further derivation, we have to express the poloidal and toroidal magnetic field by the scalars S and T using the definitions in eq. (2.25), which leads to

$$\left[-\boldsymbol{r} \times \operatorname{grad} T\right]_{-}^{+} = 0 \qquad \text{and} \qquad \left[-\boldsymbol{r}\Delta S + \operatorname{grad} \frac{\partial}{\partial r}(rS)\right]_{-}^{+} = 0. \tag{3.21}$$

First, we derive the boundary condition at R_σ for the toroidal part of geomagnetic field, which reads

$$-\boldsymbol{r} \times \operatorname{grad}\left(T^{+} - T^{-}\right) = 0.$$

The expression is ϑ and φ independent due to the vector product of $\boldsymbol{r}$ and the differential operator grad, which leads for a fixed $r = R_\sigma$ to a constant value:

$$\left(T^{+} - T^{-}\right) = \text{const.}$$

Considering the integral condition (2.27), we can conclude that the constant is zero, i.e.

$$T^{+} = T^{-}. \tag{3.22}$$

The boundary at R_σ is defined as a boundary between the conducting part of the Earth's mantle and the isolating part. In addition to the continuity of the toroidal geomagnetic field at R_σ, it is for any $r > R_\sigma$

$$\mu_0 \boldsymbol{j}_{\mathsf{M}} = 0 \quad \Rightarrow \quad \operatorname{rot} \boldsymbol{B}^{\mathsf{T}} = 0\,, \tag{3.23}$$

because outside of an electric conductor the current density, $\boldsymbol{j}_{\mathsf{M}}$, vanishes. In appendix D.1 is shown that from the last equation follows $\boldsymbol{B}^{\mathsf{T}} = 0$. Therefore, T^{+} is identically zero and it holds

$$T^{+} = T^{-} = 0. \tag{3.24}$$

Due to the orthonormal definition of the SHR of T in eq. (2.52), we find

$$T^{+}_{jm} = T^{-}_{jm} = 0. \tag{3.25}$$

Now, we derive the boundary condition at R_σ for the poloidal part of the geomagnetic field, which reads as follows (considering the splitting of the differential operators in eqs. (A.14) and (A.16)):

$$\left[-\boldsymbol{r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}S\right) + \frac{1}{r^2}\,\Delta_\Omega\, S\right) + \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}rS\right)\boldsymbol{e}_r + \frac{1}{r}\,\nabla_\Omega\left(\frac{\partial}{\partial r}rS\right)\right]_{-}^{+} = 0.$$

Applying all partial derivatives with respect to r leads to

$$\left[-\frac{1}{r}\left(\Delta_\Omega\, S\right)\boldsymbol{e}_r + \frac{1}{r}\,\nabla_\Omega\left(\frac{\partial}{\partial r}rS\right)\right]_{-}^{+} = 0\,.$$

Due to the orthogonality of the both contributions, both have separately to fulfill the continuity condition:

$$\left[\Delta_\Omega\, S\right]_{-}^{+} = 0 \qquad \text{and} \qquad \left[\nabla_\Omega\left(\frac{\partial}{\partial r}rS\right)\right]_{-}^{+} = 0 \tag{3.26}$$

The first condition, which reads like

$$\Delta_\Omega\left(S^{+} - S^{-}\right) = 0,$$

leads for a fixed $r = R_\sigma$ with the condition in eq. (2.26) and the resultant condition, $\int\limits_\Omega \Delta_\Omega\, S \mathrm{d}\Omega = 0$, to

$$S^{+} = S^{-}. \tag{3.27}$$

For the second condition, we apply the partial derivative with respect to r,

$$\nabla_\Omega \Big(S^+ - S^- \Big) + r\, \nabla_\Omega \Big(\frac{\partial}{\partial r} S^+ - \frac{\partial}{\partial r} S^- \Big) = 0, \tag{3.28}$$

where the first term is identically zero by the condition in eq. (3.27), and for a fixed $r = R_\sigma$ the second term in the brackets is constant. Taking into account the condition in eq. (2.26), which also holds for $\frac{\partial}{\partial r} S$ due to the r-independent surface integral, it follows

$$\frac{\partial}{\partial r} S^+ = \frac{\partial}{\partial r} S^-. \tag{3.29}$$

With the orthonormality of the SHR of S in eq. (2.43), we can conclude

$$S^+_{jm} = S^-_{jm} \qquad \text{and} \qquad \frac{\partial}{\partial r} S^+_{jm} = \frac{\partial}{\partial r} S^-_{jm}. \tag{3.30}$$

Compared with eqs. (B.26) to (B.28), these conditions correspond with the continuity of the radial and tangential components of $\boldsymbol{B}^{\mathsf{P}}$, respectively.

3.2.2 Boundary conditions at the core-mantle boundary $r = R_{\mathsf{CMB}}$

Like for the boundary conditions at $r = R_\sigma$, their derivation at the CMB is also based on the continuity of the geomagnetic field $\boldsymbol{B}$. Therefore, eq. (3.20) is also valid for the CMB at $r = R_{\mathsf{CMB}}$ (see fig. 3.1). For the toroidal geomagnetic field and the related field-generating scalar T, we find like in section 3.2.1

$$T^+ = T^-. \tag{3.31}$$

In contrast to the derivation before, the CMB is a boundary between two conducting domains, where the conductivity is discontinuous. Therefore, the toroidal geomagnetic field exists on both sides and do not vanish like in eq. (3.24). Using the same argument of the orthogonality of the SHR of T it follows

$$T^+_{jm} = T^-_{jm}. \tag{3.32}$$

The derivation of the boundary conditions for the poloidal geomagnetic field is completely the same like their derivation in section 3.2.1. Hence, eqs. (3.27),(3.29) and (3.30) are also valid at the CMB. For the SHR of the field-generating scalar S holds for $r = R_{\mathsf{CMB}}$:

$$S^+_{jm} = S^-_{jm} \qquad \text{and} \qquad \frac{\partial}{\partial r} S^+_{jm} = \frac{\partial}{\partial r} S^-_{jm}.$$

At the CMB, we need for the computation of the toroidal geomagnetic field in the mantle either a complete solution of the core induction equation (3.8) delivering $T(R_{\mathsf{CMB}})$ or an additional boundary condition, that is given by the continuity of the tangential component of the electric field, $\boldsymbol{r} \times \boldsymbol{E}$. Due to the lack of a suitable model of the Earth's dynamo based on eqs. (3.16) and (3.17), we use the latter, which reads

$$\begin{aligned} \big[\boldsymbol{r} \times \boldsymbol{E}\big]^+_- &= 0, \\ \boldsymbol{r} \times \big(\boldsymbol{E}^+ - \boldsymbol{E}^-\big) &= 0. \end{aligned} \tag{3.33}$$

Now, we express the electric field by the current density, $\boldsymbol{j}$, using eq. (3.4) and considering that E^+ is related to the current density in the mantle, given in eq. (3.5), and $\boldsymbol{E}^-$ by the current density in the fluid outer core, given by eq. (3.7). In addition, we take into account the Maxwell equation (3.1) which leads to

$$\boldsymbol{r} \times \left[\frac{1}{\mu_0 \sigma_{\mathsf{M}}} \operatorname{rot} \boldsymbol{B}^+ - \left(\frac{1}{\mu_0 \sigma_{\mathsf{C}}} \operatorname{rot} \boldsymbol{B}^- - \big(\boldsymbol{u} \times \boldsymbol{B}^-\big) - \boldsymbol{E}^{\mathsf{e}} \right) \right] = 0.$$

For the further derivation, we split this condition into its toroidal and poloidal contributions. All field quantities are expressed by the related field-generating scalars defined in eqs. (2.25) and (3.11)–(3.14). The toroidal part reads

$$\boldsymbol{r} \times \left[\frac{1}{\mu_0 \sigma_{\text{M}}} \operatorname{rot} \operatorname{rot} \big(\boldsymbol{r} T^{+} \big) - \frac{1}{\mu_0 \sigma_{\text{C}}} \operatorname{rot} \operatorname{rot} \big(\boldsymbol{r} T^{-} \big) + \boldsymbol{r} V + \operatorname{grad} W + \boldsymbol{r} V^{\text{e}} + \operatorname{grad} W^{\text{e}} \right] = 0.$$

We apply the relation in eq. (2.28) to $\operatorname{rot}\operatorname{rot}(\boldsymbol{r}T)$ and consider that the vector product $\boldsymbol{r} \times \boldsymbol{r}A$ vanishes identically for any scalar A:

$$\boldsymbol{r} \times \operatorname{grad} \left(\frac{1}{\sigma_{\text{m}}} \left[\frac{\partial}{\partial r} (rT) \right]^{+} - \frac{1}{\sigma_{\text{C}}} \left[\frac{\partial}{\partial r} (rT) \right]^{-} + \mu_0 W + \mu_0 W^{\text{e}} \right) = 0. \tag{3.34}$$

To fulfill the condition in eq. (3.34) at the CMB, $r = R_{\text{CMB}}$, the expression in the braces has to be constant. Due to the general representation of $(\boldsymbol{u} \times \boldsymbol{B})$ in eqs. (3.11)–(3.12), it is only possible to set up two normation conditions, which are given in eq (3.15) for V and U. The field-generating scalars W and W^{e} are only determined except for arbitrary integration constants. It is possible to choose this additive constants, that the following second boundary condition for T holds:

$$\frac{1}{\sigma_{\text{C}}} \left[\frac{\partial}{\partial r} (rT) \right]^{-} - \frac{1}{\sigma_{\text{M}}} \left[\frac{\partial}{\partial r} (rT) \right]^{+} = \mu_0 \big(W + W^{\text{e}} \big). \tag{3.35}$$

We introduce further approximations mentioned above to reduce the boundary condition and derive a boundary value of the third kind for T. First, we assume that W^{e} is identically zero, which means we neglect the contribution due to the turbulent flow. This corresponds with the restriction of our investigation on decadal time scales. Turbulent contributions would variate on much shorter time and spatial scales than considered here. Moreover, we assume

$$\frac{\sigma_{\text{M}}}{\sigma_{\text{C}}} \left[\frac{\partial}{\partial r} (rT) \right]^{-} \cong F\,,$$

which leads to

$$\left[\frac{\partial}{\partial r} (rT) \right]^{+} \cong -\mu_0 \sigma_{\text{M}} W + F. \tag{3.36}$$

This simplification is based on the conception, that F is determined to a large extend by the toroidal geomagnetic dynamo field, which variates very slowly with time, whereas the variations on the decadal time scale considered here, do not contribute to F. This assumption is analogous to the conventional separation of $\boldsymbol{B}^{\text{P}}$ into a main and secular variation field, whereas the different sources of the geomagnetic field are related to the specific field (dynamo processes to the main field, CMB surface flow to the secular variation field). Therefore, we neglect F for the further investigation. With these assumption, we can only determine the time-variable part of T. A time-independent part of T, related to the geomagnetic dynamo field can not be determined by this approach. Consequently, constant differences between EM and necessary mechanic torques should be eliminated (see section 4.3).

Following this line of argumentation, the second boundary condition for the field-generating scalar T specifies a boundary value of the third kind:

$$\left[\frac{\partial}{\partial r} (rT) \right]^{+} = -\mu_0 \sigma_{\text{M}} W. \tag{3.37}$$

Due to the orthogonality of the SHR, we can also conclude that the following relation holds:

$$\left[\frac{\partial}{\partial r} (rT_{jm}) \right]^{+} = -\mu_0 \sigma_{\text{M}} W_{jm}. \tag{3.38}$$

3.2.3 Initial-boundary value problem for the field-generating scalar T

In this section, we summarize the initial-boundary value problem for the toroidal geomagnetic field in a electrically conducting mantle, represented by the SHR of the field-generating scalar T. In section 3.1, the induction equation for the toroidal geomagnetic field is presented and its related SHR is given in eq. (3.18). In section 3.2.1 and 3.2.2, the related boundary conditions are derived (eqs. (3.25), (3.32) and (3.38)).

To obtain a compact form of the governing partial differential equation in eq. (3.18), we introduce the following definitions:

$$\Phi := \left[\frac{2}{r} - \frac{1}{\sigma_{\mathsf{M}}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{\mathsf{M}}(r)\right], \tag{3.39}$$

$$\Theta := \left[\frac{j(j+1)}{r^2} + \frac{1}{r\sigma_{\mathsf{M}}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{\mathsf{M}}(r)\right], \tag{3.40}$$

$$\Psi := \mu_0\sigma_{\mathsf{M}}(r). \tag{3.41}$$

Moreover, we use a reduced notation for the partial derivatives and neglect all arguments of the field-generating scalar, according to:

$$\mathsf{T} := T_{jm}(r,t), \tag{3.42}$$

$$\mathsf{T}_{,r} := \frac{\partial}{\partial r}T_{jm}(r,t), \tag{3.43}$$

$$\mathsf{T}_{,rr} := \frac{\partial^2}{\partial r^2}T_{jm}(r,t), \tag{3.44}$$

$$\mathsf{T}_{,t} := \frac{\partial}{\partial t}T_{jm}(r,t). \tag{3.45}$$

$$\tag{3.46}$$

The initial-boundary value problem (IBVP) for the field-generating scalar T in the SHR for the toroidal geomagnetic field in an electrically conducting mantle is set up by the differential equation

$$\mathsf{T}_{,rr} + \Phi\,\mathsf{T},r - \Theta\,\mathsf{T} - \Psi\,\mathsf{T}_{,t} = 0 \tag{3.47}$$

and the boundary conditions

$$\text{at} \quad r = R_\sigma \qquad \mathsf{T}^+ = \mathsf{T}^- = 0, \tag{3.48}$$

$$\text{at} \quad r = R_{\mathsf{CMB}} \qquad \mathsf{T}^+ = \mathsf{T}^- \quad \text{and} \quad \left[(r\mathsf{T})_{,r}\right]^+ = -\mu_0\sigma_{\mathsf{M}}W_{jm}. \tag{3.49}$$

In addition, we need to prescribe an initial value for $T_{jm}(r,t=0)$. We discuss this problem in section 3.4.2, where we also present the numerical methods, which we apply to solve the IBVP.

3.3 Calculation of the field-generating scalar W for a divergence free velocity field at the CMB

For the calculation of the field-generating scalar W, we use a more special relation, derived from the defining one in eq. (3.12). It is given by

$$\Delta_\Omega W = \boldsymbol{r}\cdot\mathrm{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]. \tag{3.50}$$

The equivalence of this expressions is shown in appendix D.3. For the further derivation, we use the SHR of W presented in appendix D.3,

$$W_{jm} = \frac{-1}{j(j+1)}\int\limits_\Omega \boldsymbol{r}\cdot\mathrm{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]Y^*_{jm}(\Omega)\,\mathrm{d}\Omega. \tag{3.51}$$

Here, Y^*_{jm} denotes the complex conjugate of the SH base function. In the following, the basic idea is to represent all quantities of the right-hand side by SH and use known relations between SH base functions, their partial derivatives and the Clebsch-Gordan coefficients. The detailed derivation is presented in appendices D.3 and D.4, which leads to

$$W_{jm} = \frac{-1}{j(j+1)} \sum_{klst} k(k+1) S_{kl}(t) \left[\mathbf{L}^{jm}_{klst}\, P_{st}(t) - \mathbf{K}^{jm}_{klst}\, Q_{st}(t) \right], \tag{3.52}$$

where the coupling coefficients are given by

$$\begin{aligned} \mathbf{K}^{jm}_{klst} &= \frac{1}{2}\, [k(k+1) - s(s+1) - j(j+1)] \\ &\quad \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2j+1)}}\, \mathbf{C}^{j\,0}_{k\,0\,s\,0}\, \mathbf{C}^{j\,m}_{k\,l\,s\,t}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} \mathbf{L}^{jm}_{klst} &= \frac{i}{2} \sqrt{(k+s+j+2)(k+s-j)(k-s+j+1)(-k+s+j+1)} \\ &\quad \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2j+3)}}\, \mathbf{C}^{(j+1)\,0}_{k\,0\,s\,0}\, \mathbf{C}^{j\,m}_{k\,l\,s\,t}. \end{aligned} \tag{3.54}$$

Here, $\mathbf{C}^{j\,m}_{k\,l\,s\,t}$, denotes the Clebsch-Gordan coefficients according to their definition in Varshalovich et al. (1989, Chap. 8). The relations between the coupling coefficients $\mathbf{K}^{jm}_{klst}$ and $\mathbf{L}^{jm}_{klst}$ in eqs. (3.53) and (3.54) and the integral kernel in eq. (3.51) are derived in detail in appendix D.4.

3.4 Solving the BVP for the toroidal magnetic field

The basic idea for the solution of the BVP for the toroidal geomagnetic field in the mantle is to express all derivatives in the describing differential equation by finite differences. This method can be numerically implemented quite straight forward, which is presented in chapter 4. The reformulation of the BVP in the simplified quasi-stationary case is summarized in section 3.4.1 and for the general case in section 3.4.2.

3.4.1 Solving the BVP for the quasi-stationary case

First, we focus on the assumption of the quasi-stationary case. That means we neglect the Ψ-term (3.41) in the describing differential equation (3.47). This assumption has different motivations: (i) to study the most simple case of the induction equation, (ii) to be consistent with the assumptions made for the determination of the fluid flow in the outer core and (iii) to provide a method to calculate an initial value for the toroidal geomagnetic field through the whole electrically conducting part of the Earth's mantle.

The partial differential in eq. (3.47) reads with the assumption of the quasi-stationary case

$$\mathsf{T}_{,rr} + \Phi\, \mathsf{T}_{,r} - \Theta\, \mathsf{T} = 0\,, \tag{3.55}$$

which is reduced to an ordinary differential equation of second order in r. Due to the quasi-stationary case, only the boundary values W are time-dependent. We follow the common approach of finite differences (e.g. Ciarlet & Lions, 1990, Chap. I.1) to solve the ordinary differential equation (3.55). Therefore, we introduce only a discretization for the radial direction:

$$r_i = R_{\text{CMB}} + i\,\Delta r \qquad \text{for} \quad i = 0, \ldots, i_{\text{max}}\,, \tag{3.56}$$

where

$$r_0 = R_{\text{CMB}} \qquad \text{and} \qquad r_{i_{\text{max}}} = R_\sigma\,. \tag{3.57}$$

With the notation

$$\mathsf{T}_i = T_{jm}(r_i)\,, \tag{3.58}$$

we find for the partial derivatives with respect to r the approximation by the finite differences

$$\mathsf{T}_{i,r} \cong \frac{1}{2\,\Delta r}\big(\mathsf{T}_{i+1} - \mathsf{T}_{i-1}\big) \tag{3.59}$$

$$\mathsf{T}_{i,rr} \cong \frac{1}{(\Delta r)^2}\big(\mathsf{T}_{i+1} - 2\mathsf{T}_i + \mathsf{T}_{i-1}\big)\,. \tag{3.60}$$

This leads to the approximation of the reduced differential equation (3.55) by

$$\frac{1}{(\Delta r)^2}\big(\mathsf{T}_{i+1} - 2\mathsf{T}_i + \mathsf{T}_{i-1}\big) + \Phi_i\,\frac{1}{2\,\Delta r}\big(\mathsf{T}_{i+1} - \mathsf{T}_{i-1}\big) - \Theta_i\,\mathsf{T}_i = 0\,. \tag{3.61}$$

Here, the label i at Φ_i and Θ_i denotes that the related expressions in eqs. (3.39) and (3.40) are calculated at $r = r_i$, respectively. Moreover, this equation is only valid for $i \in [1, i_{\text{max}} - 1]$. For the special cases $i = 0$ and $i = i_{\text{max}}$, we have to consider the boundary conditions of the BVP. The case for $i = i_{\text{max}}$ is related to the boundary condition (3.48) and the second relation in (3.57), i. e. we can conclude

$$\mathsf{T}_{i_{\text{max}}} = 0\,. \tag{3.62}$$

The second boundary condition in (3.49) reads in the reduced notation by applying the partial derivative with respect to r

$$\begin{aligned}\mathsf{T}_0 + r_0\mathsf{T}_{0,r} &= -\mu_0\sigma_{\text{M}}W_{jm}\,,\\ \mathsf{T}_{0,r} &= \frac{-1}{R_{\text{CMB}}}\big(\mu_0\sigma_{\text{M}}W_{jm} + \mathsf{T}_0\big)\,.\end{aligned} \tag{3.63}$$

Using eq. (3.63), we find an expression for T_{-1}, which is needed for the discrete differential equation for $i = 0$ due to the quadratic approximation by eqs. (3.59) and (3.60):

$$\mathsf{T}_{-1} = \frac{2\Delta r}{R_{\text{CMB}}}\big(\mu_0\sigma_{\text{M}}W_{jm} + \mathsf{T}_0\big) + \mathsf{T}_1\,. \tag{3.64}$$

With this expression, we can set up the discrete differential equation for $i = 0$ from eq. (3.61) as follows

$$\frac{2}{(\Delta r)^2}\big(\mathsf{T}_1 - \mathsf{T}_0\big) + \frac{2}{\Delta r\,R_{\text{CMB}}}\,\big(\mu_0\sigma_{\text{M}}W_{jm} + \mathsf{T}_0\big) + \Phi_0\left(\frac{1}{\Delta r}\mathsf{T}_1 - \frac{1}{R_{\text{CMB}}}\big(\mu_0\sigma_{\text{M}}W_{jm} + \mathsf{T}_0\big)\right) - \Theta_0\mathsf{T}_0 = 0\,. \tag{3.65}$$

The discretization of the ordinary differential equation for the quasi-stationary case of the BVP is realized by eqs. (3.61), (3.62) and (3.65), which also consider the boundary conditions. This equations set up a system of linear equations for T_i and can be rewritten in a matrix notation using the definition

$$\mathbf{T} = [\mathsf{T}_0, \ldots, \mathsf{T}_{i_{\text{max}}}]^\top\,, \tag{3.66}$$

which leads to

$$\mathbb{A}\,\mathbf{T} = \mathbf{R}\,. \tag{3.67}$$

The matrix $\mathbb{A}$ and the related vector of the right-hand sides are defined in appendix D.6 in eqs. (D.61) and (D.62), respectively. On this formulation is based the implementation of the BVP, which is presented together with an example in section 4.1.

3.4.2 Solving the BVP for the time-dependent case

In the general case, we have to consider the time-dependence of the partial differential equation (3.47) of the IBVP, which reads

$$\mathsf{T}_{,rr} + \Phi\,\mathsf{T},r - \Theta\,\mathsf{T} = \Psi\,\mathsf{T}_{,t}\,.$$

Beside the spatial discretization in eq. (3.56), we introduce now the discretization of the time by

$$t^n = t^0 + n\Delta t \qquad \text{for} \quad n = 0, \ldots, n_{\text{max}}\,, \tag{3.68}$$

where the time interval for the observed geomagnetic field is given by

$$[t^0, t^{n_{\text{max}}}]. \tag{3.69}$$

We choose the so-called Crank-Nicolson approach (e.g. Ciarlet & Lions, 1990, Chap. I.2) for a quadratic approximation in time and space to solve the IBVP. The spatial derivatives are approximated by eqs. (3.59) and (3.60) and the time derivative is set up as average of an implicit and explicit scheme, which reads

$$\begin{aligned}\frac{\mathsf{T}_i^{n+1} - \mathsf{T}_i^n}{\Delta t} &= \frac{1}{2(\Delta r)^2 \Psi_i}\Big(\mathsf{T}_{i+1}^{n+1} - 2\mathsf{T}_i^{n+1} + \mathsf{T}_{i-1}^{n+1} + \mathsf{T}_{i+1}^{n} - 2\mathsf{T}_i^{n} + \mathsf{T}_{i-1}^{n}\Big) \\ &+ \frac{\Phi_i}{4\Delta r \Psi_i}\Big(\mathsf{T}_{i+1}^{n+1} - \mathsf{T}_{i-1}^{n+1} + \mathsf{T}_{i+1}^{n} - \mathsf{T}_{i-1}^{n}\Big) - \frac{\Theta_i}{2\Psi_i}\Big(\mathsf{T}_i^{n+1} + \mathsf{T}_i^{n}\Big),\end{aligned} \tag{3.70}$$

where we have used the reduced notation

$$\mathsf{T}_i^n = T_{jm}(r_i, t^n). \tag{3.71}$$

This discretization of the partial differential equation is only valid for $i \in [1, i_{\text{max}} - 1]$ and we have again to consider the boundary conditions for the special cases $i = 0$ and $i = i_{\text{max}}$. Moreover, eq. (3.70) is only a relation between the solution for T_i^n throughout the spatial domain and the solution for the next time step, T_i^{n+1}. We have to tackle two different problems. First, we have to determine an initial value, T_i^0, throughout the spatial domain, and second, we have to set up a scheme, to realize a quadratic approximation in time.

As mentioned in section 3.4.1, one reason to consider the quasi-stationary case is, to obtain a time-independent solution, which will be used as initial value in the general case. For the given boundary value at $t = t^0$, we compute the quasi-stationary solution $\mathbf{T}^0$ for the field-generating scalar of the toroidal geomagnetic field, where

$$\mathbf{T}^n = [\mathsf{T}_0^n, \ldots, \mathsf{T}_{i_{\text{max}}}^n]^\top \tag{3.72}$$

defines the time-dependent vector of the field-generating scalars.

Moreover, we have to set up the Crank-Nicolson approach to realize the quadratic approximation of the time derivatives. In contrast to the spatial derivatives, which are discretized by a quadratic approximation, in the Crank-Nicolson approach we set up a two-step scheme for the time derivative: (i) by eq. (3.70) we compute the solution T_i^{n+1} based on the result T_i^n and (ii) we calculate the average mid-point solution by $\mathsf{T}_i^{n+\frac{1}{2}} = (\mathsf{T}_i^{n+1} + \mathsf{T}_i^n)/2$. This average solution is then used as the 'old' solution in eq. (3.70). More efficiently is this approach described in the matrix notation. Before we can set up a system of equations, analogous to the derivation in section 3.4.1, we have to formulate the discretized differential equation for the special cases $i = 0$ and $i = i_{\text{max}}$ using the related boundary conditions. This is completely analogous to the derivation in section 3.4.1 and we end up with the expressions in eqs. (3.62) and (3.65). We have only to consider the new notation, which expresses the time dependence by the superscript n. In appendix D.7 are given the related matrix $\mathbb{A}$ and the vector of the right-hand sides $\mathbf{R}^n$, which is now time-dependent. We can write the Crank-Nicolson approach then in the matrix notation as follows:

$$\mathbb{A}\,\mathbf{T}^{n+1} = \mathbf{R}^n, \tag{3.73}$$

where the vector of right-hand sides is a function of $\mathbf{T}^n$. The details are given in eqs. (D.71) and (D.73). Moreover, we have to apply the averaging in time to determine the right-hand side for the next step by

$$\mathbf{T}^{n+\frac{1}{2}} = \frac{1}{2}\big(\mathbf{T}^{n+1} + \mathbf{T}^n\big). \tag{3.74}$$

Therefore, the solution is determined by the Crank-Nicolson approach first for the next time step t^{n+1} by eq. (3.73), and then approximated at a half time step back ($t^{n+\frac{1}{2}}$) according to eq. (3.74). Hereby a quadratic approximation in time is realized. The implementation of the IBVP is based on this formulation, which is presented together with an example in section 4.2.

Implementation of the BVP and the calculation of the EM torques | 4

Beside the theoretical description of the BVP in chapter 3, we present here the numerical implementation of the BVP for the toroidal magentic field at the CMB. We consider first the quasi-stationary case, second the time-dependent case and in the third section we present an example for the computation of the EM coupling torques.

For the computation of the toroidal geomagnetic field at the CMB, we need the SHR of the poloidal geomagnetic field and the fluid-flow velocity in the outer core at the CMB. Both quantities are used here as prescribed values and we refer to the publications shown below, which present solutions to these problems. Ballani et al. (2002) developed the non-harmonic downward continuation (NHDC) to compute the poloidal geomagnetic field at the CMB, taking into account a radial conductivity profile for the Earth's mantle, $\sigma_{\text{M}}(r)$. We chose here for the examples in this chapter a profile, shown in Figure 4.1 and prescribed by

$$\sigma_{\text{M}}(r) = \begin{cases} c & \text{for} \quad R_{\text{CMB}} < r < R_{\text{s}} \\ c\exp(1)\exp\left(\dfrac{-1}{1-((R_{\text{s}}-r)/s)^2}\right) & \text{for} \quad R_{\text{s}} \leq r \leq R_\sigma \\ 0 & \text{for} \quad r > R_\sigma \end{cases} , \tag{4.1}$$

where

$$R_{\text{CMB}} = 3485\,\text{km}, R_{\text{s}} = 3670\,\text{km}, R_\sigma = 3694\,\text{km}, s = 24\,\text{km}, c = 10^3\ \text{Sm}^{-1}\,.$$

The NHDC provides the Gauss coefficients at the CMB, which are combined by eq. (E.6) to the SHR coefficients S_{jm}. For the required geomagnetic field model at the Earth's surface, used as input for the NHDC, we chose the model C^{3}FM from Wardinski & Holme (2006).

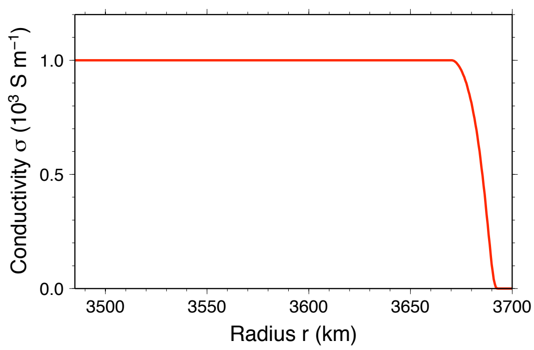

Figure 4.1: Electric conductivity profile, which is used for the examples in this chapter. The related function is given in eq. (4.1)

Moreover, we need to know the fluid-flow velocity, $\boldsymbol{u}$, in the outer core at the CMB. Wardinski (2004) has set up a fluid-flow inversion approach, where the time variable poloidal geomagnetic field is the main input. For details of this method, we refer to this work as well as to Wardinski et al. (2008).

For the implementation of the BVP for the toroidal geomagnetic field at the CMB, we need to calculate the time-dependent boundary value function $W(t)$. Its SHR coefficients W_{jm} are given in eq. (3.52), where the coupling coefficients $\mathbf{K}^{jm}_{klst}$ and $\mathbf{L}^{jm}_{klst}$ in eqs. (3.53) and (3.54) are represented by Clebsch-Gordan coefficients. Those are computed by the subroutines `cle0` and `cle`, written by Zdeněk Martinec according to the related recursion formulae in Varshalovich et al. (1989, Sec. 8.5–8.6). The quasi-stationary and the time-dependent case require each a specific approach for the implementation, which are summarized in the following sections 4.1 and 4.2.

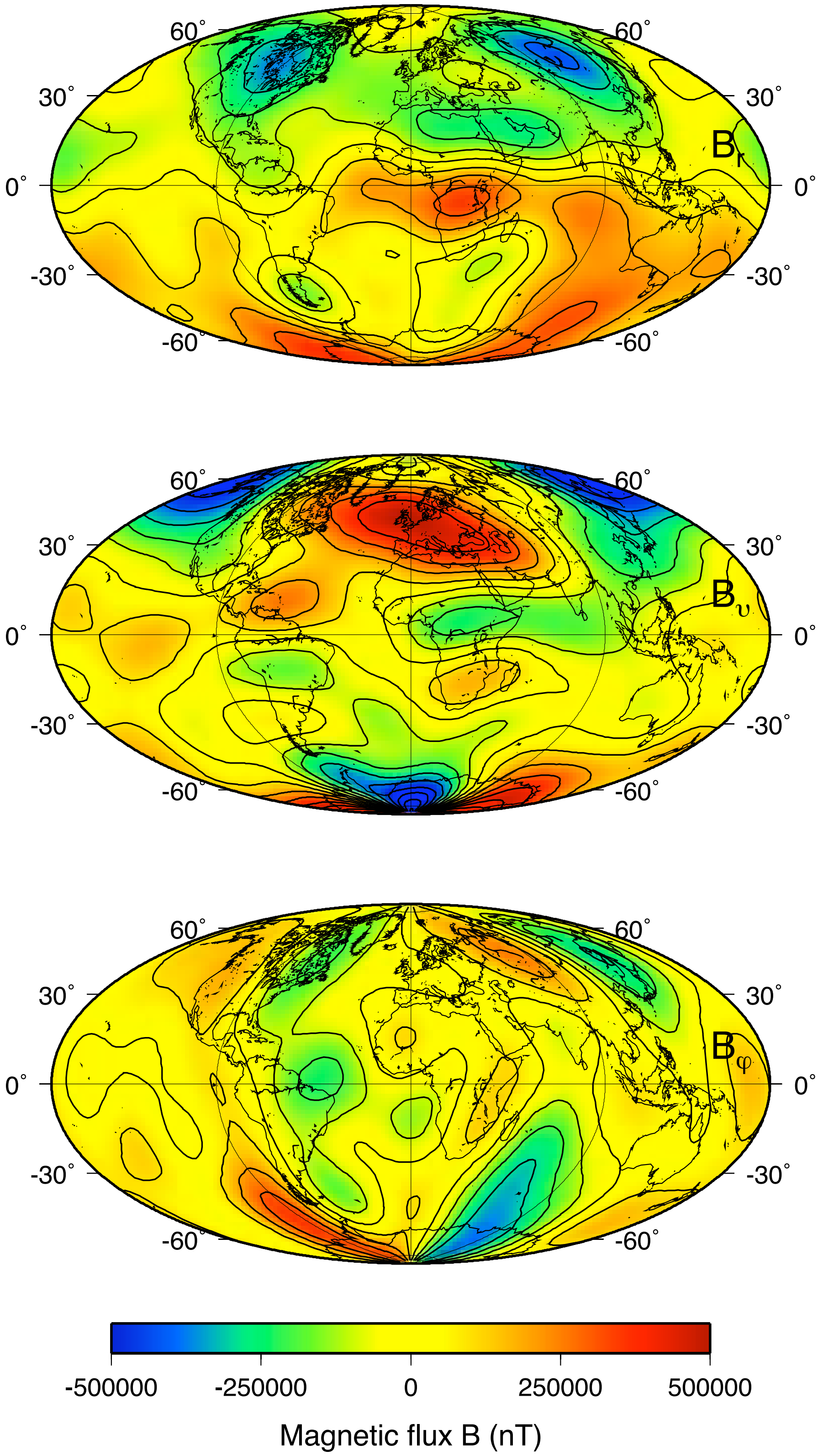

Figure 4.2: Components of the poloidal geomagnetic field for the calendar year 1990.0 at the CMB. From top to bottom: B_r^{P}, B_ϑ^{P} and B_φ^{P}.

4.1 Implementation of the BVP for the quasi-stationary case

In eq. (3.67) is given the set of equations in matrix notation, which describe the BVP in the quasi-stationary case. The matrix of the system in eq. (D.61) has a tri-diagonal structure. For each vector of

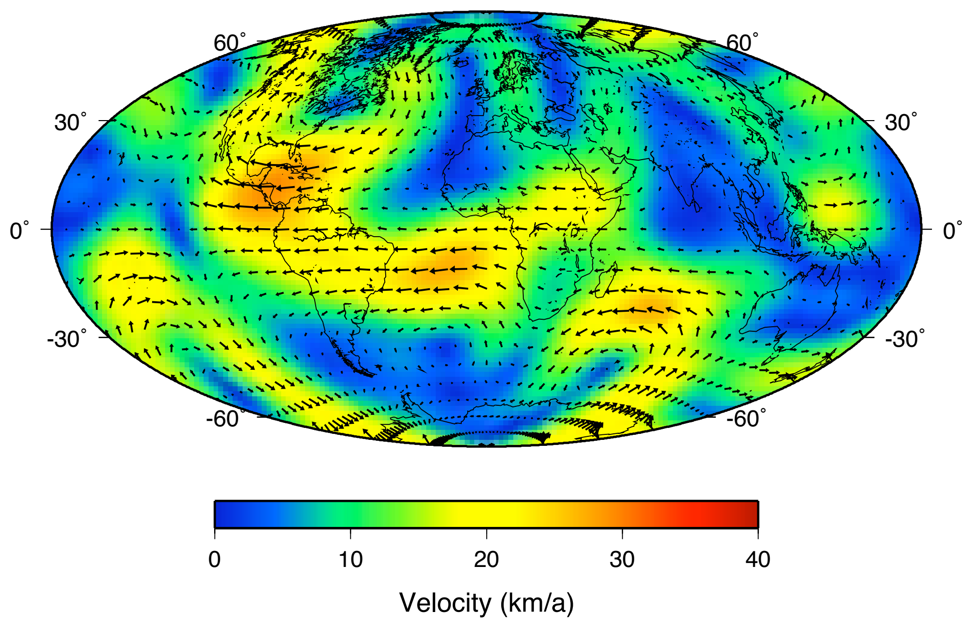

Figure 4.3: Fluid-flow velocity field in the outer core for the calendar year 1990.0 at the CMB. The colour map shows the amplitude of the horizontal velocity and the arrows indicate the direction.

the right-hand side in eq. (D.62), which contains the time-dependent boundary value W_{jm}, the solution is obtained by a special elimination approach. This is implemented into the subroutine **tridag**, taken from Press et al. (1992, Sec. 2.4).

We are now able to calculate the SHR of the field-generating scalar $T_{jm}(t)$ for each time, based onto the prescribed field-generating scalars $S_{jm}(t)$, $P_{jm}(t)$ and Q_{jm}, representing the poloidal geomagnetic field and the horizontal fluid-flow velocity at the CMB, respectively. As an example, considering the conductivity profile (4.1), we present the related field quantities. Figure 4.2 shows the three components of the poloidal geomagnetic field for the calendar year 1990.0, calculated by the NHDC at the CMB. Here, the dipole structure of the r-component is not as dominate as at the Earth's surface, what is caused by the NHDC. The components of poloidal geomagnetic field are in the order of $\pm\,500\,000$ nT, which is about seven times larger than at the Earth's surface.

Moreover, the fluid-flow velocity of the outer core at the CMB is required, which is presented in Figure 4.3. This velocity field is computed by the fluid-flow inversion, as mentioned before, and is provided by Ingo Wardinski. The amplitude of the horizontal velocity reaches 30 km a^{-1}. Beside this high velocity eddies, it is clearly to see a westerly flow along the equator, which has a mean velocity of about 10 km a^{-1}. In addition to the input quantities, we show in Figure 4.4 the resulting toroidal geomagnetic field. Due to the fact, that B_r is sole poloidal, only the angular components are presented. The spatial pattern in both components of the toroidal geomagnetic field are completely different from the related of the poloidal field. Moreover, the toroidal field components are in the order of $\pm\,70\,000$ nT, which is much smaller than the poloidal. We have to emphasize that by the chosen boundary value in eq. (3.38) is only considered the varying, $\boldsymbol{u}$-dependent contribution to the field-generating scalar T. Therefore, the resulting toroidal field represents only this contribution and we neglect some undetermined constant contribution to the toroidal field.

4.2 Implementation of the BVP for the time-dependent case

The eqs. (3.73) and (3.74) describe the Crank-Nicolson approach by a set of equations in matrix notation and an averaging of the solution vector, respectively. The related matrix of the system is given in eq. (D.71). In (D.73) is presented the relation to compute the vector of the right-hand sides. Solving the system of equations leads to the solution for the next discrete time step, whereas the related vector of

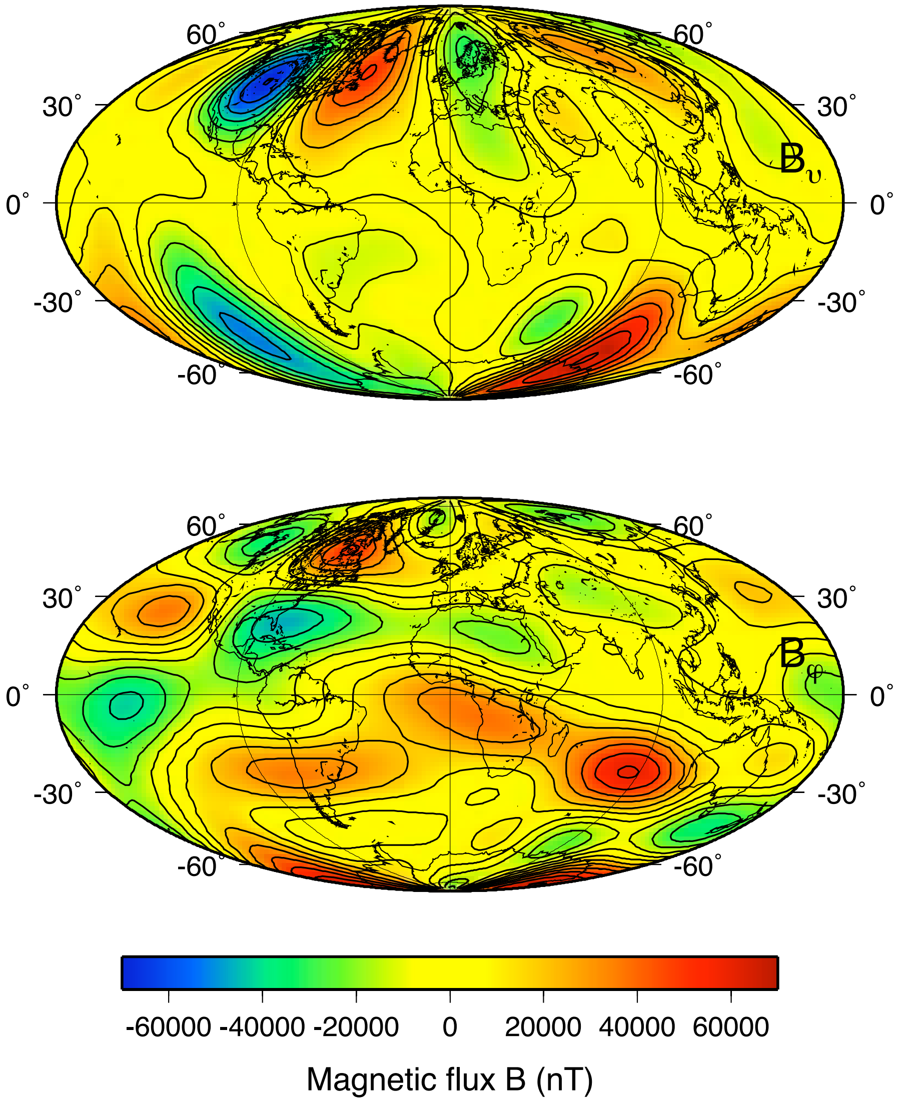

Figure 4.4: Components of the toroidal geomagnetic field for the calendar year 1990.0 at the CMB. From top to bottom: $B^{\mathrm{T}}_{\vartheta}$ and B^{T}_{φ}.

right-hand sides is not only depending on the time-dependent boundary value $W_{jm}(t)$ but also on the solution of the time step before. One consequence is that we have to prescribe an initial value for the solution vector **T** for the first time step. Beside the most simple choice of a zero vector, we use the solution of the quasi-stationary case. Numerically, we have only to set up a robust method for the solution of the set of equations in (3.73). The matrix $\mathbb{A}$ is ill-conditioned, which requires a more robust technique than before in the quasi-stationary case. We apply here a LU-decomposition with partial pivoting, which is performed once at the begin, and the related back substitution, realized in the subroutines **ludcmp** and **lubksb** taken from Press et al. (1992, Sec. 2.3).

In summary, for the Crank-Nicolson approach we have to perform first the solving of the system of equation for related right-hand sides, and in a second step the averaging of the solution, according to eq. (D.73). With this two-step approach, we only advance by a half time step. So, we have to perform numerically two times the number of steps as in the quasi-stationary case for the same time series of boundary values.

For the numerical example, we chose the same input quantities as in the quasi-stationary case (conductivity profile, poloidal geomagnetic field and fluid-flow velocity field at the CMB). Therefore, we present in Figure 4.5 only the resulting toroidal geomagnetic field, again for the calendar year 1990.0 at the CMB. A comparison of the toroidal geomagntic field components at the CMB for the quasi-stationary

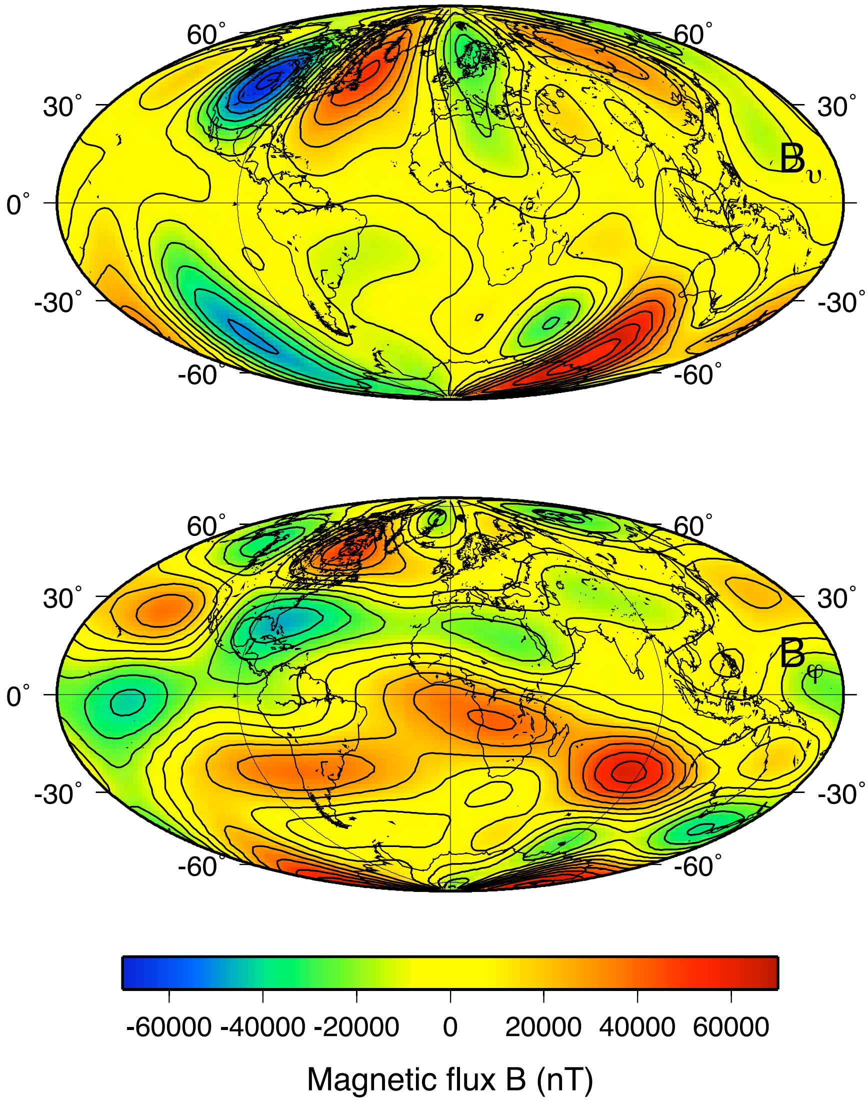

Figure 4.5: Components of the toroidal geomagnetic field for the calendar year 1990.0 at the CMB. From top to bottom: $B^{\mathrm{T}}_{\vartheta}$ and B^{T}_{φ}.

(Fig. 4.4) and the time-dependent (Fig. 4.5) case shows only small differences in the maximal amplitudes, whereas the spatial patterns are similar. This differences, which reach up to $10\,\%$ of the absolute values, are due to the time-dependent formulation of the BVP, which emphasize the importance to consider the time-dependent case in the determination of the toroidal geomagnetic field.

4.3 Implementation of the computation of the EM coupling torques

In section 2.4, the analytical expressions for the axial and non-axial EM coupling torques are summarized. For the poloidal and toroidal axial EM torques, we find in eqs. (2.51) and (2.55) the related summation formulae, respectively. The poloidal and toroidal non-axial torques are combined in the complex expressions, given in eqs. (2.56) and (2.72). For these complex combinations, the related summation formulae are given by eqs. (2.71) and (2.79). All these equations are only dependent on the SHR of the field-generating scalars S and T at the CMB. Therefore, the EM coupling torque computation reduces to the summation of these coefficients according to the related equation above, and the related transformation of the SH coefficients in the normalization used in geomagnetism (Schmidt's) and the orthonormal

SHR used here.

The used geomagnetic field model C^{3}FM from Wardinski & Holme (2006) is expressed by coefficients up to maximum degree $j_{\text{max}} = 15$. We consider only degrees and orders up to $j_{\text{max}} = 8$ to ensure that only the contribution of the core-generated geomagnetic field enter into the torque computation (Greiner-Mai et al., 2007). In Figure 4.6, we compare the variation of the components of the EM coupling torques based on the solution of the BVP for the toroidal geomagnetic field for quasi-stationary and time-dependent case. The variation of each component of the EM coupling torque is determined by detrending. Beside the similar time-behavior in all components with only small differences in the extrema, we observe a time lag between both results. The results based on the time-dependent formulation of the BVP for the toroidal geomagnetic field at the CMB seems to be shifted by approximately six month. Due to the time-dependent formulation, the EM torque has to act at the CMB 'later' than in the quasi-stationary case, to be consistent with the same observed geomagnetic field at the Earth's surface.

We conclude from this simple calculation that it is quite important to consider the time-dependent case in the determination of the geomagnetic field at the CMB for an electrically conducting mantle. The presented approach allows us to calculate the EM coupling torques for realistic, but only radially dependent conductivity profiles of the Earth's mantle.

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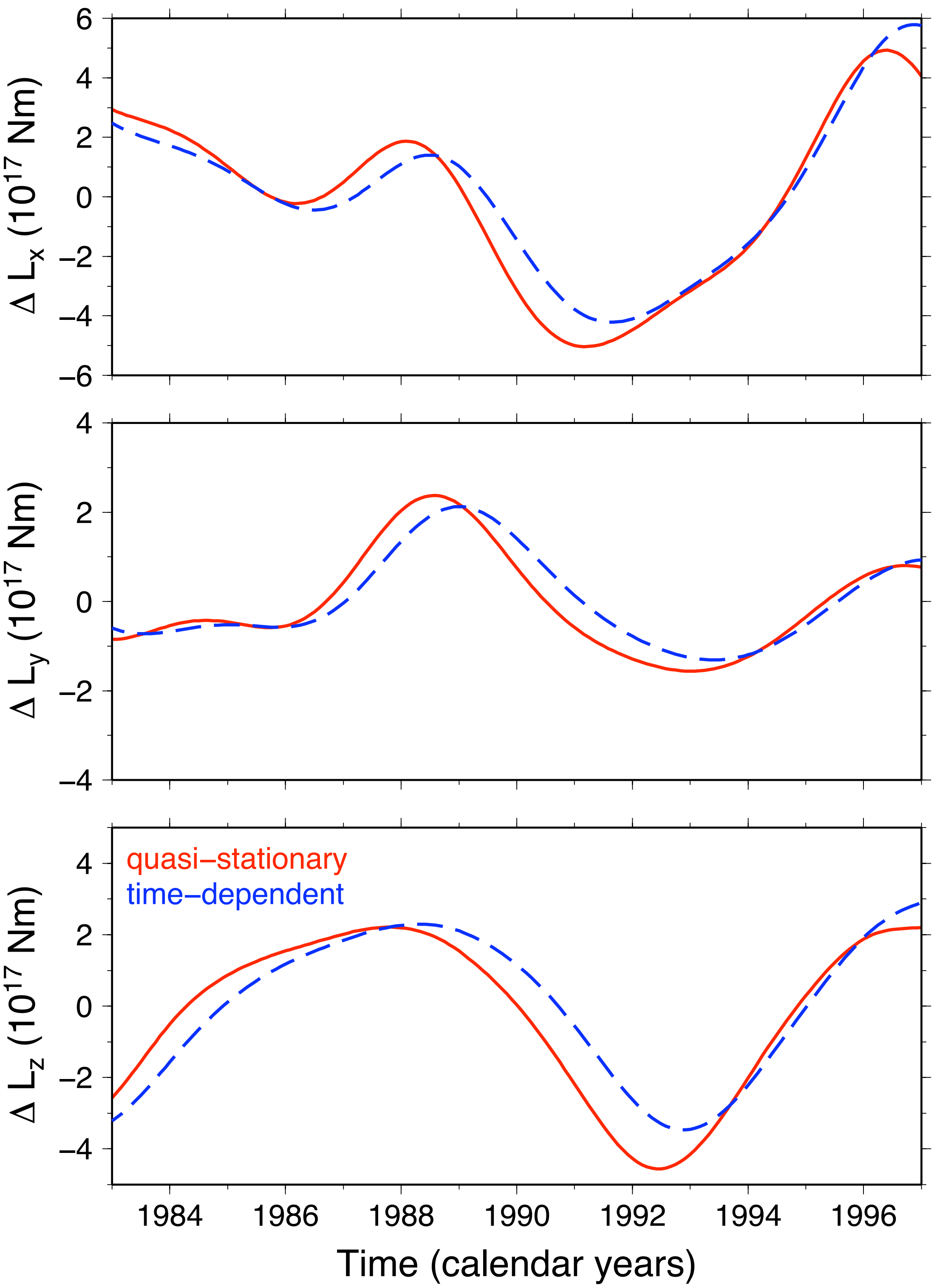

Figure 4.6: Time series of the components of the EM coupling torques for the quasi-stationary and time-dependent case. From top to bottom are shown: ΔL_x, ΔL_y and ΔL_z, respectively.

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Spherical harmonics and related derivations | A

A.1 Definition of scalar spherical harmonics and basic relations

In this section, we shortly summarize the definition and basic relations of spherical harmonics (SH). The most definitions are according to Varshalovich et al. (1989, Chap. 5) and we follow their notation, where Y_{jm} are the scalar spherical harmonics of degree j and order m, which are defind by:

$$Y_{jm}(\Omega) := P_{jm}(\cos\vartheta)e^{im\varphi}. \tag{A.1}$$

Here are $i = \sqrt{-1}$, $\Omega = (\vartheta, \varphi)$ and P_{jm} the associated Legendre functions, which are defined as follows:

$$P_{jm}(\cos\vartheta) := (-1)^m \sqrt{\frac{2j+1}{4\pi}\frac{(j-m)!}{(j+m)!}}(\sin\vartheta)^m \frac{d^m}{(d\cos\vartheta)^m} P_j(\cos\vartheta), \tag{A.2}$$

$$P_j(\cos\vartheta) := \frac{1}{2^j j!}\frac{d^j(\cos^2\vartheta - 1)^j}{d(\cos\vartheta)^j}. \tag{A.3}$$

For the degree and order of the SH and Legendre functions the co-domains

$$j = 0, 1, 2, 3, \ldots, \infty, \tag{A.4}$$

$$m = -j, \ldots, 0, \ldots, j \tag{A.5}$$

are valid. The SH are orthonormal on the unit sphere (Varshalovich et al., 1989, Chap. 5):

$$\int\limits_{\Omega_0} Y_{jm}(\Omega) Y^*_{j'm'}(\Omega)\, \mathrm{d}\Omega = \delta_{jj'}\delta_{mm'}. \tag{A.6}$$

Here, $*$ denotes the complex conjugate and δ_{ij} is Kronecker's symbol. For the complex conjugate SH the following relation holds:

$$Y^*_{jm}(\Omega) = (-1)^m Y_{j\,-m}(\Omega). \tag{A.7}$$

Using the definitions and relations above, we can derive explicitly the SH for different degrees, j, and order, m. For degrees $j = 0, 1, 2$ and the related order m, the SH are given by:

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \tag{A.8}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\vartheta, \tag{A.9}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{i\varphi}, \tag{A.10}$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\vartheta - 1\right), \tag{A.11}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\vartheta\cos\vartheta\, e^{i\varphi}, \tag{A.12}$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\vartheta\, e^{i2\pi}. \tag{A.13}$$

We summarize here also the differential operators in spherical coordinates and the related splitting into angular and radial parts, which are given for the Nabla operator by

$$\nabla = \left[\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\nabla_\Omega\right], \tag{A.14}$$

$$\nabla_\Omega = \left[\frac{\partial}{\partial\vartheta}\boldsymbol{e}_\vartheta + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi\right], \tag{A.15}$$

and for the Laplace operator by

$$\Delta = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \Delta_\Omega\right], \tag{A.16}$$

$$\Delta_\Omega = \left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial}{\partial\varphi^2}\right] \tag{A.17}$$

The SH are eigenfunctions of the angular part of the Lapace operator so that

$$\Delta_\Omega\, Y_{jm}(\Omega) = -j(j+1)Y_{jm}(\Omega) \tag{A.18}$$

is valid.

Moreover, we show here the used derivative of SH with respect to the spherical coordinates φ and ϑ (Varshalovich et al., 1989, Sec. 5.8):

$$\frac{\partial}{\partial\varphi}Y_{jm}(\Omega) = imY_{jm}(\Omega), \tag{A.19}$$

$$\begin{aligned}\frac{\partial}{\partial\vartheta}Y_{jm}(\Omega) &= \frac{1}{2}\sqrt{j(j+1)-m(m+1)}\,Y_{j\,(m+1)}(\Omega)e^{-i\varphi}\\ &\quad - \frac{1}{2}\sqrt{j(j+1)-m(m-1)}\,Y_{j\,(m-1)}(\Omega)e^{i\varphi},\end{aligned} \tag{A.20}$$

$$\begin{aligned}\sin\vartheta\frac{\partial}{\partial\vartheta}Y_{jm}(\Omega) &= j\sqrt{\frac{(j+1)^2-m^2}{(2j+1)(2j+3)}}\,Y_{(j+1)\,m}(\Omega)\\ &\quad - (j+1)\sqrt{\frac{j^2-m^2}{(2j+1)(2j-1)}}\,Y_{(j-1)\,m}(\Omega).\end{aligned} \tag{A.21}$$

A.2 Vector spherical harmonics

The vector spherical harmonics we have chosen here are defined as follows (Varshalovich et al., 1989)

$$\boldsymbol{S}^{(-1)}_{jm}(\Omega) = \boldsymbol{e}_r Y_{jm}(\Omega)\,, \tag{A.22}$$

$$\boldsymbol{S}^{(+1)}_{jm}(\Omega) = \nabla_\Omega\, Y_{jm}(\Omega)\,, \tag{A.23}$$

$$\boldsymbol{S}^{(0)}_{jm}(\Omega) = \mathrm{L}_\Omega\, Y_{jm}(\Omega)\,, \tag{A.24}$$

where $\boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta$ and $\boldsymbol{e}_\varphi$ are the spherical base vectors and the differential operator ∇_Ω is given in eq. (A.15). The operator L_Ω is defined by

$$\mathrm{L}_\Omega = \boldsymbol{e}_r \times \nabla_\Omega = \left[\frac{\partial}{\partial\vartheta}\boldsymbol{e}_\varphi - \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\vartheta\right]. \tag{A.25}$$

Two vector spherical harmonics $\boldsymbol{S}^{(\lambda)}_{jm}(\Omega)$ and $\boldsymbol{S}^{(\lambda')}_{j'm'}(\Omega)$, with different degree, $j \neq j'$, and different order, $m \neq m'$, as well as different indices, $\lambda \neq \lambda'$, are orthogonal, expressed by

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(\lambda)}_{jm}(\Omega)\cdot\left[\boldsymbol{S}^{(\lambda')}_{j'm'}(\Omega)\right]^*\,\mathrm{d}\Omega = 0, \tag{A.26}$$

where the dot denotes the scalar product of vectors. Moreover, we summarize the following expressions for orthogonal vector spherical harmonics:

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(-1)}_{jm}(\Omega)\cdot\left[\boldsymbol{S}^{(-1)}_{j'm'}(\Omega)\right]^*\,\mathrm{d}\Omega = \delta_{jj'}\delta_{mm'}, \tag{A.27}$$

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(1)}_{jm}(\Omega)\cdot\left[\boldsymbol{S}^{(1)}_{j'm'}(\Omega)\right]^*\,\mathrm{d}\Omega = j(j+1)\delta_{jj'}\delta_{mm'}, \tag{A.28}$$

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(0)}_{jm}(\Omega)\cdot\left[\boldsymbol{S}^{(0)}_{j'm'}(\Omega)\right]^*\,\mathrm{d}\Omega = j(j+1)\delta_{jj'}\delta_{mm'}. \tag{A.29}$$

For the vector spherical harmonics, as defined here, we can easily derive the following expressions:

$$\boldsymbol{S}^{(\lambda)}_{j\,-m}(\Omega) = (-1)^m \left[\boldsymbol{S}^{(\lambda)}_{jm}\right]^*, \tag{A.30}$$

$$\boldsymbol{e}_r \times \boldsymbol{S}^{(0)}_{jm}(\Omega) = -\boldsymbol{S}^{(1)}_{jm}(\Omega), \tag{A.31}$$

$$\boldsymbol{e}_r \times \boldsymbol{S}^{(-1)}_{jm}(\Omega) = 0, \tag{A.32}$$

$$\boldsymbol{e}_r \times \boldsymbol{S}^{(1)}_{jm}(\Omega) = \boldsymbol{S}^{(0)}_{jm}(\Omega). \tag{A.33}$$

A.3 Relation between representations in real and complex spherical harmonics

With the definition of the complex SH in eq. (A.1) follows for a scalar function S the representation

$$S(r,\Omega) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_{jm}(r) Y_{jm}(\Omega). \tag{A.34}$$

In the literature, it is also common to use the real spherical harmonic representation of a scalar function, which is given by

$$S(r,\Omega) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left[S^{\mathrm{c}}_{kl}(r)\cos(l\varphi) + S^{\mathrm{s}}_{kl}(r)\sin(l\varphi)\right] \tilde{P}_{kl}(\cos\vartheta), \tag{A.35}$$

where $\tilde{P}_{kl}$ is the Legendre function in Ferrers-Neumann normalization,

$$\tilde{P}_{kl}(\cos\vartheta) = (\sin\vartheta)^l \frac{d^l}{(d\cos\vartheta)^l} P_k(\cos\vartheta), \tag{A.36}$$

and P_k are the Legendre polynominals defined in eq. (A.3). The orthogonality is expressed by

$$\int_0^{\pi} \tilde{P}_{kl}(\cos\vartheta)\tilde{P}_{k'l'}(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta = \frac{2}{2k+1}\frac{(k+l)!}{(k-l)!}\,\delta_{kk'}\,\delta_{ll'}\,, \tag{A.37}$$

in contrast to the orthogonality for the associated Legendre functions defined in eq. (A.2)

$$\int_0^{\pi} P_{kl}(\cos\vartheta)P_{k'l'}(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta = \frac{1}{2\pi}\delta_{kk'}\,\delta_{ll'}\,. \tag{A.38}$$

We find the following relation between the Legendre function in the different normalization,

$$\tilde{P}_{kl}(\cos\vartheta) = \lambda_{kl} P_{kl}(\cos\vartheta), \tag{A.39}$$

where λ_{kl} is given by

$$\lambda_{kl} = (-1)^l \sqrt{\frac{4\pi}{2k+1}\frac{(k+l)!}{(k-l)!}}. \tag{A.40}$$

We compare now both expressions for S to find a relation between the real and the complex coefficients. First, we split the summation in eq. (A.34), which yields

$$\begin{aligned} S(r,\Omega) &= \sum_{j=0}^{\infty}\Bigg[S_{j\,0}(r)Y_{j\,0}(\Omega) + \sum_{m=-j}^{-1} S_{jm}(r)Y_{jm}(\Omega) + \sum_{m=1}^{j} S_{jm}(r)Y_{jm}(\Omega)\Bigg], \\ &= \sum_{j=0}^{\infty}\Bigg[S_{j\,0}(r)Y_{j\,0}(\Omega) + \sum_{m=1}^{j}\big(S_{j\,-m}(r)Y_{j\,-m}(\Omega) + S_{jm}(r)Y_{jm}(\Omega)\big)\Bigg]. \end{aligned}$$

With the definition of the complex conjugated in eq. (A.7) it is

$$S(r,\Omega) = \sum_{j=0}^{\infty}\Big[S_{j\,0}(r)Y_{j\,0}(\Omega) + \sum_{m=1}^{j}\big((-1)^m S_{j\,-m}(r)Y^*_{jm}(\Omega) + S_{jm}(r)Y_{jm}(\Omega)\big)\Big].$$

For a real scalar function S, the following relation holds:

$$(-1)^m S_{j\,-m}(r) = S^*_{jm}, \tag{A.41}$$

from which we derive

$$\begin{aligned} S(r,\Omega) &= \sum_{j=0}^{\infty}\Big[S_{j\,0}(r)Y_{j\,0}(\Omega) + \sum_{m=1}^{j}\big(S^*_{jm}(r)Y^*_{jm}(\Omega) + S_{jm}(r)Y_{jm}(\Omega)\big)\Big], \\ &= \sum_{j=0}^{\infty}\Big[S_{j\,0}(r)Y_{j\,0}(\Omega) + \sum_{m=1}^{j} 2\,\mathrm{Re}\big(S_{jm}(r)Y_{jm}(\Omega)\big)\Big]. \end{aligned}$$

Using the definition of the scalar SH, we find

$$\begin{aligned} S(r,\Omega) = \sum_{j=0}^{\infty}\Bigg\{ & S_{j\,0}(r)\cos(0)P_{jm}(\cos\vartheta) + \sum_{m=1}^{j} 2\,\Big[\mathrm{Re}\,(S_{jm}(r))\cos(m\varphi)P_{jm}(\cos\vartheta) \\ & - \mathrm{Im}\,(S_{jm}(r))\sin(m\varphi)P_{jm}(\cos\vartheta)\Big]\Bigg\}, \\ = & \sum_{j=0}^{\infty}\sum_{m=1}^{j}\frac{(2-\delta_{m0})}{\lambda_{jm}}\Big[\mathrm{Re}\big(S_{jm}(r)\big)\cos(m\varphi) - \mathrm{Im}\big(S_{jm}(r)\big)\sin(m\varphi)\Big]\tilde{P}_{jm}(\cos\vartheta). \end{aligned} \tag{A.42}$$

The comparison of eq. (A.42) and eq. (A.35) yields the relation between the real and complex coefficients of the SH representation for $m \geq 0$:

$$\begin{Bmatrix} S^{\mathrm{c}}_{jm}(r) \\ S^{\mathrm{s}}_{jm}(r) \end{Bmatrix} = \frac{(2-\delta_{m0})}{\lambda_{jm}}\begin{Bmatrix} \mathrm{Re}\big(S_{jm}(r)\big) \\ -\mathrm{Im}\big(S_{jm}(r)\big) \end{Bmatrix}. \tag{A.43}$$

With that, we can derive the relation vice versa for $m \geq 0$:

$$S_{jm}(r) = \frac{\lambda_{jm}}{(2-\delta_{m0})}\big(S^{\mathrm{c}}_{jm}(r) - iS^{\mathrm{s}}_{jm}(r)\big). \tag{A.44}$$

For $m < 0$ we can calculate the coefficients by eq. (A.41).

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Additional derivations for the EM torque formulation | B

B.1 Magnetic stress tensor

Here, we derive the divergence of the magnetic stress tensor using the definition of the divergence of a tensor

$$\operatorname{div}\boldsymbol{\mathcal{M}} = \overset{3}{\boldsymbol{\mathcal{E}}}\,[\operatorname{grad}\boldsymbol{\mathcal{M}}]\,\boldsymbol{\mathcal{I}}\,, \tag{B.1}$$

where the Levi-Civita tensor is defined by the extension of the Levi-Civita symbol, ϵ_{klm}, by

$$\overset{3}{\boldsymbol{\mathcal{E}}} = \left(\epsilon_{klm}\boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_m\right) \tag{B.2}$$

and $\boldsymbol{\mathcal{I}}$ is the identity tensor. With that, we find using Einstein's summation convention

$$\begin{aligned}\operatorname{div}\boldsymbol{\mathcal{M}} &= \mathcal{M}_{kl,\,m}\left(\boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_m\right)\left(\boldsymbol{e}_j \otimes \boldsymbol{e}_j\right),\\ &= \mathcal{M}_{kl,\,m}\delta_{lj}\,\delta_{mj}\,\boldsymbol{e}_k,\\ &= \mathcal{M}_{kj,\,j}\,\boldsymbol{e}_k,\end{aligned}$$

where the indices following a comma denotes the partial derivative with respect to the related base vector. With the definition in eq. (2.6) it follows

$$= \frac{1}{\mu_0}\left(B_j\frac{\partial}{\partial x_j}B_k + \underbrace{B_k\frac{\partial}{\partial x_j}B_j}_{=\,0} - \frac{1}{2}\frac{\partial}{\partial x_k}(B_nB_n)\right)\,\boldsymbol{e}_k. \tag{B.3}$$

B.2 Symmetry property of a tensor of second order

In the derivation of the surface integral of the EM coupling torque, we need the following property of a symmetric tensor of second order, $\boldsymbol{\mathcal{M}}$ (see eq. (2.11)):

$$\boldsymbol{r} \times \operatorname{div}\boldsymbol{\mathcal{M}} = \operatorname{div}\big(\boldsymbol{r} \times \boldsymbol{\mathcal{M}}\big).$$

To prove this property, we use the more general relation for any tensor of second order, $\boldsymbol{\mathcal{T}}$, and a vector, $\boldsymbol{v}$:

$$\operatorname{div}\big(\boldsymbol{v} \times \boldsymbol{\mathcal{T}}\big) = \boldsymbol{v} \times \operatorname{div}\boldsymbol{\mathcal{T}} + \operatorname{grad}\boldsymbol{v} \times \boldsymbol{\mathcal{T}}. \tag{B.4}$$

With the Levi-Civita tensor and the definition of the vector product between a vector, $\boldsymbol{v}$, and a tensor, $\boldsymbol{\mathcal{T}}$ by

$$\boldsymbol{v} \times \boldsymbol{\mathcal{T}} = \left[\overset{3}{\boldsymbol{\mathcal{E}}}\big(\boldsymbol{v} \otimes \boldsymbol{\mathcal{T}}\big)\right]^{\underline{2}}, \tag{B.5}$$

where $\underline{2}$ indicates that the result is a tensor of second order, it follows in Einstein's summation convention

$$\begin{aligned}\boldsymbol{v} \times \boldsymbol{\mathcal{T}} &= \left[\big(\epsilon_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k\big)\big(v_r\boldsymbol{e}_r \otimes \mathcal{T}_{st}\boldsymbol{e}_s \otimes \boldsymbol{e}_t\big)\right]^{\underline{2}},\\ &= \epsilon_{ijk}v_r\mathcal{T}_{st}\delta_{jr}\,\delta_{ks}\boldsymbol{e}_i \otimes \boldsymbol{e}_t,\\ &= \epsilon_{ijk}v_j\mathcal{T}_{kt}\boldsymbol{e}_i \otimes \boldsymbol{e}_t,\end{aligned}$$

and we find

$$\begin{aligned}\operatorname{div}\big(\boldsymbol{v}\times\boldsymbol{\mathcal{T}}\big) &= \epsilon_{ijk}\big(v_j\mathcal{T}_{kt}\big)_{,s}\big(\boldsymbol{e}_i\otimes\boldsymbol{e}_t\otimes\boldsymbol{e}_s\big)\big(\boldsymbol{e}_n\otimes\boldsymbol{e}_n\big),\\ &= \epsilon_{ijk}\big(v_j\mathcal{T}_{kt}\big)_{,s}\delta_{tn}\,\delta_{sn}\boldsymbol{e}_i,\\ &= \epsilon_{ijk}\big(v_j\mathcal{T}_{kn}\big)_{,n}\boldsymbol{e}_i,\\ &= \epsilon_{ijk}v_j\mathcal{T}_{kn\,,n}\boldsymbol{e}_i+\epsilon_{ijk}v_{j\,,n}\mathcal{T}_{kn}\boldsymbol{e}_i.\end{aligned}\tag{B.6}$$

From the first term of the right-hand side of eq. (B.4) we can derive the expression

$$\begin{aligned}\boldsymbol{v}\times\operatorname{div}\boldsymbol{\mathcal{T}} &= \left[\overset{3}{\boldsymbol{\mathcal{E}}}\big(v_s\boldsymbol{e}_s\big)\otimes\big(\mathcal{T}_{ln\,,n}\boldsymbol{e}_n\big)\right],\\ &= \big(\epsilon_{ijk}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\big)\big(v_s\boldsymbol{e}_s\otimes\mathcal{T}_{ln\,,n}\boldsymbol{e}_l\big),\\ &= \epsilon_{ijk}v_s\mathcal{T}_{ln\,,n}\delta_{js}\,\delta_{kl}\boldsymbol{e}_i,\\ &= \epsilon_{ijk}v_j\mathcal{T}_{kn\,,n}\boldsymbol{e}_i,\end{aligned}\tag{B.7}$$

which is identically with the first part of eq. (B.6). Moreover, the second part of eq. (B.4) yields

$$\begin{aligned}\operatorname{grad}\boldsymbol{v}\times\boldsymbol{\mathcal{T}} &= \overset{3}{\boldsymbol{\mathcal{E}}}\big(\operatorname{grad}\boldsymbol{v}\boldsymbol{\mathcal{T}}^{\top}\big),\\ &= \big(\epsilon_{ijk}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\big)\big(v_{l\,,n}(\boldsymbol{e}_l\otimes\boldsymbol{e}_n)\mathcal{T}_{st}(\boldsymbol{e}_t\otimes\boldsymbol{e}_s)\big),\\ &= \epsilon_{ijk}v_{l\,,n}\mathcal{T}_{st}\delta_{nt}\,\delta_{jl}\,\delta_{ks}\,\boldsymbol{e}_i,\\ &= \epsilon_{ijk}v_{j\,,n}\mathcal{T}_{kn}\boldsymbol{e}_i,\end{aligned}\tag{B.8}$$

which is identically with the second part of eq. (B.6) and the whole relation is proved. Now we apply eq. (B.4) to the position vector, $\boldsymbol{r}$, and the magnetic stress tensor, $\boldsymbol{\mathcal{M}}$. To prove the relation in eq. (2.11), we have to show that the second term in eq. (B.4) vanishs for a symmetric tensor like the magnetic stress tensor:

$$\begin{aligned}\operatorname{grad}\boldsymbol{r}\times\boldsymbol{\mathcal{M}} &= \big(r_{l\,,m}\boldsymbol{e}_l\otimes\boldsymbol{e}_m\big)\times\big(\mathcal{M}_{st}\boldsymbol{e}_s\otimes\boldsymbol{e}_t\big),\\ &= \big(\delta_{lm}\boldsymbol{e}_l\otimes\boldsymbol{e}_m\big)\times\big(\mathcal{M}_{st}\boldsymbol{e}_s\otimes\boldsymbol{e}_t\big),\\ &= \left(\overset{3}{\boldsymbol{\mathcal{E}}}\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{M}}^{\top}\right),\\ &= \big(\epsilon_{ijk}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\big)\mathcal{M}_{st}\big(\boldsymbol{e}_t\otimes\boldsymbol{e}_s\big),\\ &= \epsilon_{ijk}\delta_{jt}\,\delta_{ks}\mathcal{M}_{st}\boldsymbol{e}_i,\\ &= \epsilon_{ijk}\mathcal{M}_{kj}\boldsymbol{e}_i.\end{aligned}$$

With the symmetry of $\boldsymbol{\mathcal{M}}$ follows

$$\mathcal{M}_{kj}=\mathcal{M}_{jk}$$

and therefore

$$\operatorname{grad}\boldsymbol{r}\times\boldsymbol{\mathcal{M}}=0.\tag{B.9}$$

B.3 Derivation of the eq. (2.9)

For the derivation of eq. (2.9), the definition of the rotation of an arbitrary vector $\boldsymbol{T}$ is needed:

$$\operatorname{rot}\boldsymbol{T}=\overset{3}{\boldsymbol{\mathcal{E}}}\left[\operatorname{grad}\boldsymbol{T}^{\top}\right].\tag{B.10}$$

Using Einstein's summation convention, this definition reads

$$
\begin{aligned}
\operatorname{rot} \boldsymbol{B} \times \boldsymbol{B} &= \overset{3}{\boldsymbol{\mathcal{E}}} \left[\operatorname{grad} \boldsymbol{B}^{\top}\right] \times \boldsymbol{B}, \\
&= \left(\epsilon_{ijn} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_n\right) B_{o,p} \left(\boldsymbol{e}_p \otimes \boldsymbol{e}_o\right) \times \left(B_s \boldsymbol{e}_s\right), \\
&= \epsilon_{ijn} B_{o,p} \delta_{jp} \delta_{no} \boldsymbol{e}_i \times \left(B_s \boldsymbol{e}_s\right), \\
&= \left(\epsilon_{ijn} B_{n,j} \boldsymbol{e}_i\right) \times \left(B_s \boldsymbol{e}_s\right), \\
&= \overset{3}{\boldsymbol{\mathcal{E}}} \left[\epsilon_{ijn} B_{n,\,j} \boldsymbol{e}_i\right] \otimes \left(B_s \boldsymbol{e}_s\right), \\
&= \left(\epsilon_{klm} \boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_m\right) \left(\epsilon_{ijn} B_{n,\,j} \boldsymbol{e}_i \otimes B_s \boldsymbol{e}_s\right), \\
&= \epsilon_{klm} \epsilon_{ijn} B_{n,\,j} B_s \delta_{li} \delta_{ms} \boldsymbol{e}_k, \\
&= \epsilon_{ijn} \epsilon_{imk} B_{n,\,j} B_m \boldsymbol{e}_k \,.
\end{aligned}
$$

With the relation

$$
\epsilon_{ijn} \epsilon_{imk} = \delta_{jm} \delta_{nk} - \delta_{jk} \delta_{nm} \tag{B.11}
$$

follows

$$
\begin{aligned}
\operatorname{rot} \boldsymbol{B} \times \boldsymbol{B} &= \left(\delta_{jm} \delta_{nk} - \delta_{jk} \delta_{nm}\right) B_{n,\,j} B_m \, \boldsymbol{e}_k, \\
&= B_{k,j} B_j \boldsymbol{e}_k - B_{n,k} B_n \boldsymbol{e}_k, \\
&= B_j \frac{\partial}{\partial x_j} B_k \boldsymbol{e}_k - B_n \frac{\partial}{\partial x_k} B_n \boldsymbol{e}_k, \\
&= \left(B_j \frac{\partial}{\partial x_j} B_k - \frac{1}{2} \frac{\partial (B_n B_n)}{\partial x_k}\right) \boldsymbol{e}_k \,.
\end{aligned} \tag{B.12}
$$

B.4 Derivation of the surface integral in eq. (2.14)

For the electromagnetic coupling torque $\boldsymbol{L}$, we found in eq. (2.12) the surface integral

$$
\boldsymbol{L} = \int\limits_{\Omega} \left(\boldsymbol{r} \times \boldsymbol{\mathcal{M}}\right) \cdot \boldsymbol{n} \, r^2 \, \mathrm{d}\Omega.
$$

With the definition of the vector product between a vector and a tensor, it is

$$
\begin{aligned}
\left(\boldsymbol{r} \times \boldsymbol{\mathcal{M}}\right) \cdot \boldsymbol{n} &= \left[\overset{3}{\boldsymbol{\mathcal{E}}} \left(\boldsymbol{r} \otimes \boldsymbol{\mathcal{M}}\right)\right]^{\underline{2}} \cdot \boldsymbol{n}, \\
&= \left[\overset{3}{\boldsymbol{\mathcal{E}}} \left(r_s \boldsymbol{e}_s \otimes \mathcal{M}_{jm} \boldsymbol{e}_j \otimes \boldsymbol{e}_m\right)\right]^{\underline{2}} \cdot \boldsymbol{n},
\end{aligned}
$$

and with the definition of the magnetic stress tensor in eq. (2.6) follows in Einstein's summation condition

$$
\begin{aligned}
&= \left[\left(\epsilon_{pqr} \boldsymbol{e}_p \otimes \boldsymbol{e}_q \otimes \boldsymbol{e}_r\right) r_s \boldsymbol{e}_s \otimes \left(\frac{1}{\mu_0} \left(B_j B_m - \frac{1}{2} B_k B_k \delta_{jm}\right) \boldsymbol{e}_j \otimes \boldsymbol{e}_m\right)\right]^{\underline{2}} \cdot \boldsymbol{n}, \\
&= \left[\epsilon_{pqr} r_s \frac{1}{\mu_0} \left(B_j B_m - \frac{1}{2} B_k B_k \delta_{jm}\right) \delta_{qs} \, \delta r j \boldsymbol{e}_p \otimes \boldsymbol{e}_m\right] \cdot \boldsymbol{n}, \\
&= \frac{1}{\mu_0} \left[\epsilon_{pqr} \left(r_q B_r B_m - \frac{1}{2} B_k B_k \delta_{rm}\right) \boldsymbol{e}_p \otimes \boldsymbol{e}_m\right] \cdot \boldsymbol{n}, \\
&= \frac{1}{\mu_0} \left[\left(\epsilon_{pqr} r_q B_r\right) B_m - \left(\epsilon_{pqr} \frac{1}{2} B_k B_k \delta_{rm}\right)\right] \boldsymbol{e}_p \otimes \boldsymbol{e}_m \cdot \boldsymbol{n}.
\end{aligned}
$$

Using the definition of the vector product yields

$$
\left(\boldsymbol{r} \times \boldsymbol{\mathcal{M}}\right) \cdot \boldsymbol{n} = \frac{1}{\mu_0} \left[\left(\boldsymbol{r} \times \boldsymbol{B}\right) \left(\boldsymbol{B} \cdot \boldsymbol{n}\right) - \frac{(B)^2}{2} \left(\boldsymbol{r} \times \boldsymbol{n}\right)\right], \tag{B.13}
$$

where the following relation has been applied:

$$
\begin{aligned}
\boldsymbol{r} \times \boldsymbol{n} &= r_s \boldsymbol{e}_s \times \boldsymbol{e}_n, \\
&= \overset{3}{\boldsymbol{\mathcal{E}}} \left[r_s \boldsymbol{e}_s \otimes \boldsymbol{e}_n \right], \\
&= \epsilon_{ijk} \left(\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \right) r_s \boldsymbol{e}_s \otimes \boldsymbol{e}_n, \\
&= \epsilon_{ijk} r_s \delta_{js} \, \delta_{kn} \boldsymbol{e}_i, \\
&= \epsilon_{ijn} r_j \boldsymbol{e}_i.
\end{aligned} \tag{B.14}
$$

We can now derive the surface integral in eq. (2.14):

$$
\boldsymbol{L} = \frac{1}{\mu_0} \int\limits_{\Omega} \left[\left(\boldsymbol{r} \times \boldsymbol{B} \right) \left(\boldsymbol{B} \cdot \boldsymbol{n} \right) - \frac{(B)^2}{2} \left(\boldsymbol{r} \times \boldsymbol{n} \right) \right] r^2 \, \mathrm{d}\Omega.
$$

B.5 Field-generating scalar functions

The field-generating scalar functions S and T fulfill eqs. (2.28) and (2.29). Here, we proove this relations in detail. With eq. (B.10) we find

$$
\begin{aligned}
\operatorname{rot}\big(\boldsymbol{r}T\big) &= \overset{3}{\boldsymbol{\mathcal{E}}} \big[\operatorname{grad}^{\top} \big(\boldsymbol{r}T\big) \big], \\
&= \epsilon_{ijn} \big(\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_n \big) \big(\boldsymbol{r}T \big)_{o\,,p} \big(\boldsymbol{e}_p \otimes \boldsymbol{e}_o \big), \\
&= \epsilon_{ijn} \big(\boldsymbol{r}T \big)_{o\,,p} \delta_{jp} \, \delta_{no} \, \boldsymbol{e}_i, \\
&= \epsilon_{ijn} \big(r_n T_{,j} + r_{n\,,j} T \big) \boldsymbol{e}_i,
\end{aligned}
$$

and with $r_{n\,,j} = 0$ follows

$$
\begin{aligned}
&= \epsilon_{ijn} r_n T_{,j} \boldsymbol{e}_i, && \text{(B.15)} \\
&= -\epsilon_{inj} r_n T_{,j} \boldsymbol{e}_i. && \text{(B.16)}
\end{aligned}
$$

This is identical with

$$
\operatorname{rot}\big(\boldsymbol{r}T\big) = -\boldsymbol{r} \times \operatorname{grad} T, \tag{B.17}
$$

by which eq. (2.29) is proved. With eq. (B.15) we can derive now for eq. (2.28)

$$
\begin{aligned}
\operatorname{rot}\operatorname{rot}\big(\boldsymbol{r}S\big) &= \operatorname{rot}\big(\epsilon_{ijn} r_n S_{,j} \boldsymbol{e}_i \big), \\
&= \overset{3}{\boldsymbol{\mathcal{E}}} \big[\operatorname{grad}^{\top} \big(\epsilon_{ijn} r_n S_{,j} \boldsymbol{e}_i \big) \big], \\
&= \epsilon_{stu} \big(\boldsymbol{e}_s \otimes \boldsymbol{e}_t \otimes \boldsymbol{e}_u \big) \big(\epsilon_{ijn} r_n S_{,j} \big)_{i\,,p} \big(\boldsymbol{e}_p \otimes \boldsymbol{e}_i \big), \\
&= \epsilon_{stu} \big(\epsilon_{ijn} r_n S_{,j} \big)_{i\,,p} \delta_{tp} \, \delta_{ui} \boldsymbol{e}_s, \\
&= \epsilon_{sti} \epsilon_{ijn} \big(r_n S_{,j} \big)_{i\,,t} \boldsymbol{e}_s, \\
&= \epsilon_{sti} \epsilon_{ijn} \big(r_{n\,,t} S_{,j} + r_n S_{,jt} \big)_i \boldsymbol{e}_s,
\end{aligned}
$$

and with eq. (B.11) follows

$$
\begin{aligned}
&= \big(\delta_{sj} \, \delta_{tn} - \delta_{sn} \, \delta_{tj} \big) \big(r_{n\,,t} S_{,j} + r_n S_{,jt} \big) \boldsymbol{e}_s, \\
&= r_{t\,,t} S_{,s} \boldsymbol{e}_s - r_s S_{,tt} \boldsymbol{e}_s.
\end{aligned}
$$

With the definitions of the gradient and Laplace operator, we obtain

$$
\operatorname{rot}\operatorname{rot}\big(\boldsymbol{r}S\big) = \operatorname{grad}\big(\frac{\partial}{\partial r} r S\big) - \boldsymbol{r} \Delta S, \tag{B.18}
$$

by which eq. (2.28) is proved.

B.6 Componental form of B in spherical coordinates

We start from eq. (2.30),

$$\boldsymbol{B} = -\boldsymbol{r}\Delta S + \operatorname{grad}\left(\frac{\partial}{\partial r} rS\right) - \boldsymbol{r} \times \operatorname{grad} T,$$

to derive the components of the magnetic flux $\boldsymbol{B}$ in spherical coordinates. We need for the further derivation with the position vector $\boldsymbol{r} = r\boldsymbol{e}_r$ the following differential operators in spherical coordinates (e. g. Varshalovich et al., 1989, Chap. 1):

$$\Delta S = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r} S) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta \frac{\partial}{\partial\vartheta} S) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} S, \tag{B.19}$$

$$\operatorname{grad}\left(\frac{\partial}{\partial r}(rS)\right) = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}(rS)\right)\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{\partial}{\partial r}(rS)\right)\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}(rS)\right)\boldsymbol{e}_\varphi. \tag{B.20}$$

Moreover, we need the relation

$$\boldsymbol{r} \times \operatorname{grad} T = \epsilon_{ijn} r_j T_{,n} \boldsymbol{e}_i,$$

which reads in spherical coordinates as

$$\begin{aligned}\boldsymbol{r} \times \operatorname{grad} T &= -r\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi} T\boldsymbol{e}_\vartheta + \frac{1}{r} r\frac{\partial}{\partial\vartheta} T\boldsymbol{e}_\varphi, \\ &= \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} T\boldsymbol{e}_\vartheta - \frac{\partial}{\partial\vartheta} T\boldsymbol{e}_\varphi. \end{aligned} \tag{B.21}$$

Considering the definition $\boldsymbol{r} = r\boldsymbol{e}_r$, we can now split the magnetic flux vector, $\boldsymbol{B}$, into its spherical components:

$$\begin{aligned} B_r &= -r\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r} S) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\frac{\partial}{\partial\vartheta} S) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} S\right) \\ &\quad + \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}(rS)\right), \end{aligned} \tag{B.22}$$

$$B_\vartheta = \frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{\partial}{\partial r}(rS)\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} T, \tag{B.23}$$

$$B_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}(rS)\right) - \frac{\partial}{\partial\vartheta} T. \tag{B.24}$$

Furthermore, we can simplify B_r in eq. (B.22) as follows. With

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r} S) &= \frac{1}{r}(2r\frac{\partial}{\partial r} S + r^2\frac{\partial^2}{\partial r^2}), \\ &= 2\frac{\partial}{\partial r} S + r\frac{\partial^2}{\partial r^2},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}(rS)\right) &= \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} S\right) + \frac{\partial}{\partial r} S, \\ &= 2\frac{\partial}{\partial r} S + r\frac{\partial^2}{\partial r^2},\end{aligned}$$

the expression (B.22) for B_r reduces to

$$B_r = -\frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\frac{\partial}{\partial\vartheta} S) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} S\right] = -\frac{1}{r}\,\Delta_\Omega\, S. \tag{B.25}$$

B.7 SHR of the components of the magnetic field

Starting from the equations for the vector components of $\boldsymbol{B}$ in appendix B.6, we can split them into poloidal and toroidal parts, related to the field-generating scalars, defined in eq. (2.25) and express this by the SHR of S and T, given in eqs. (2.43) and (2.52). For the poloidal part, we find the following expressions,

$$B_r^{\mathsf{P}} = -\frac{1}{r}\sum_{jm} \Delta_\Omega\big[S_{jm}(r)Y_{jm}(\Omega)\big], \tag{B.26}$$

$$B_\vartheta^{\mathsf{P}} = \frac{1}{r}\sum_{jm} \frac{\partial}{\partial\vartheta}\left[\frac{\partial}{\partial r}\big(rS_{jm}(r)\big)Y_{jm}(\Omega)\right], \tag{B.27}$$

$$B_\varphi^{\mathsf{P}} = \frac{1}{r\sin\vartheta}\sum_{jm} \frac{\partial}{\partial\varphi}\left[\frac{\partial}{\partial r}\big(rS_{jm}(r)\big)Y_{jm}(\Omega)\right], \tag{B.28}$$

for the toroidal part we find

$$B_r^{\mathsf{T}} = 0, \tag{B.29}$$

$$B_\vartheta^{\mathsf{T}} = \frac{1}{\sin\vartheta}\sum_{jm} \frac{\partial}{\partial\varphi}\big[T_{jm}(r)Y_{jm}(\Omega)\big], \tag{B.30}$$

$$B_\varphi^{\mathsf{T}} = -\sum_{jm} \frac{\partial}{\partial\vartheta}\big[T_{jm}(r)Y_{jm}(\Omega)\big]. \tag{B.31}$$

The relations between Gauss coefficients and the field generating scalar S in eq. (E.6) are only valid for positive orders, $m \geq 0$, of the SHR. Therefore, we need also for the vector components of $\boldsymbol{B}$ expressions, which consider only positive orders of the SHR. For eq. (B.26), we can apply eq. (A.18), which yields

$$B_r^{\mathsf{P}} = \frac{1}{r}\sum_{jm} j(j+1)S_{jm}(r)Y_{jm}(\Omega),$$

which leads to the decomposition for positive and negative orders:

$$B_r^{\mathsf{P}} = \frac{1}{r}\sum_{j=1}^{j_{\max}} j(j+1)\left[S_{j0}(r)Y_{j0}(\Omega) + \sum_{m=1}^{j} S_{jm}(r)Y_{jm}(\Omega) + \sum_{m=-j}^{-1} S_{jm}(r)Y_{jm}(\Omega)\right].$$

With the substitution $-m = \nu$ in the second sum over m follows

$$B_r^{\mathsf{P}} = \frac{1}{r}\sum_{j=1}^{j_{\max}} j(j+1)\left[S_{j0}(r)Y_{j0}(\Omega) + \sum_{m=1}^{j} S_{jm}(r)Y_{jm}(\Omega) + \sum_{\nu=1}^{j} S_{j\,-\nu}(r)Y_{j\,-\nu}(\Omega)\right],$$

which simplifies using eqs. (A.7) and (A.41) and substituting $\nu = m$ to

$$B_r^{\mathsf{P}} = \frac{1}{r}\sum_{j=1}^{j_{\max}} j(j+1)\left[S_{j0}(r)Y_{j0}(\Omega) + \sum_{m=1}^{j}\Big(S_{jm}(r)Y_{jm}(\Omega) + S_{jm}^*(r)Y_{jm}^*(\Omega)\Big)\right].$$

For all complex numbers $a, b \in \mathbb{C}$ it is valid:

$$ab + a^*b^* = 2\,\mathrm{Re}(ab). \tag{B.32}$$

This relation yields the final relation

$$B_r^{\mathsf{P}} = \frac{1}{r}\sum_{j=1}^{j_{\max}} j(j+1)\left[S_{j0}(r)Y_{j0}(\Omega) + 2\sum_{m=1}^{j}\mathrm{Re}\Big(S_{jm}(r)Y_{jm}(\Omega)\Big)\right], \tag{B.33}$$

where B_r^{P} is expressed only by non-negative orders m.

For the ϑ-componet of $\boldsymbol{B}^{\mathsf{P}}$ in eq. (B.27), the application of the product rule leads to the expression

$$B_\vartheta^{\mathsf{P}} = \frac{1}{r}\Bigg[\sum_{jm}\bigg(S_{jm}(r) + r\,\big(\frac{\partial}{\partial r}S_{jm}(r)\big)\bigg)\frac{\partial}{\partial\vartheta}Y_{jm}(\Omega)\Bigg],$$

which can be transformed with eq. (A.20) to

$$B_\vartheta^{\mathsf{P}} = \frac{1}{2r}\Bigg[\sum_{jm}\bigg(S_{jm}(r) + r\,\big(\frac{\partial}{\partial r}S_{jm}(r)\big)\bigg)\bigg(\sqrt{j(j+1)-m(m+1)}\,Y_{j\,(m+1)}(\Omega)\,e^{-i\varphi}$$
$$-\sqrt{j(j+1)-m(m-1)}\,Y_{j\,(m-1)}(\Omega)\,e^{i\varphi}\bigg)\Bigg].$$

Due to the requirement $m \geq 0$ for the order of the SHR, we reformulate the last equation with the substitution $-m = \nu$ and we find

$$B_\vartheta^{\mathsf{P}} = \frac{1}{2r}\sum_{j=1}^{j_{\max}}\Bigg[\bigg(S_{j0}(r) + r\,\big(\frac{\partial}{\partial r}S_{j0}(r)\big)\bigg)\sqrt{j(j+1)}\bigg(Y_{j\,1}(\Omega)\,e^{-i\varphi} - Y_{j\,-1}(\Omega)\,e^{i\varphi}\bigg)$$
$$+\sum_{m=1}^{j}\bigg(S_{jm}(r) + r\,\big(\frac{\partial}{\partial r}S_{jm}(r)\big)\bigg)\bigg(\sqrt{j(j+1)-m(m+1)}\,Y_{j\,(m+1)}(\Omega)\,e^{-i\varphi}$$
$$-\sqrt{j(j+1)-m(m-1)}\,Y_{j\,(m-1)}(\Omega)\,e^{i\varphi}\bigg)$$
$$+\sum_{\nu=1}^{j}\bigg(S_{j\,-\nu}(r) + r\,\big(\frac{\partial}{\partial r}S_{j\,-\nu}(r)\big)\bigg)\bigg(\sqrt{j(j+1)-\nu(\nu-1)}\,Y_{j\,-(\nu-1)}(\Omega)\,e^{-i\varphi}$$
$$-\sqrt{j(j+1)-\nu(\nu+1)}\,Y_{j\,-(\nu+1)}(\Omega)\,e^{i\varphi}\bigg)\Bigg].$$

Further, we simplify this expression using the relation of complex conjugated coefficients and SH given in eq. (A.7) and in eq. (A.41), respectively.

$$B_\vartheta^{\mathsf{P}} = \frac{1}{2r}\sum_{j=1}^{j_{\max}}\Bigg[\bigg(S_{j0}(r) + r\,\big(\frac{\partial}{\partial r}S_{j0}(r)\big)\bigg)\sqrt{j(j+1)}\bigg(Y_{j\,1}(\Omega)\,e^{-i\varphi} + \big(Y_{j\,1}(\Omega)\big)^*\big(e^{-i\varphi}\big)^*\bigg)$$
$$+\sum_{m=1}^{j}\bigg(S_{jm}(r) + r\,\big(\frac{\partial}{\partial r}S_{jm}(r)\big)\bigg)\bigg(\sqrt{j(j+1)-m(m+1)}\,Y_{j\,(m+1)}(\Omega)\,e^{-i\varphi}$$
$$-\sqrt{j(j+1)-m(m-1)}\,Y_{j\,(m-1)}(\Omega)\,e^{i\varphi}\bigg)$$
$$+\sum_{\nu=1}^{j}(-1)^{\nu}\bigg(S^*_{j\,\nu}(r) + r\,\big(\frac{\partial}{\partial r}S_{j\,\nu}(r)\big)^*\bigg)\bigg(\sqrt{j(j+1)-\nu(\nu-1)}$$
$$(-1)^{\nu-1}Y^*_{j\,(\nu-1)}(\Omega)\big(e^{i\varphi}\big)^* - \sqrt{j(j+1)-\nu(\nu+1)}(-1)^{\nu+1}Y^*_{j\,(\nu+1)}(\Omega)\big(e^{i\varphi}\big)^*\bigg)\Bigg].$$

For all complex numbers $a, b, c \in \mathbb{C}$, it is valid:

$$abc + a^*b^*c^* = 2\,\mathrm{Re}(abc). \tag{B.34}$$

In addition, we use eq. (B.32) and the substitution $\nu = m$ to derive

$$B^{\mathsf{P}}_{\vartheta} = \frac{1}{r}\sum_{j=1}^{j_{\max}} \Bigg[\Bigg(S_{j0}(r) + r\,\big(\frac{\partial}{\partial r} S_{j0}(r)\big) \Bigg) \sqrt{j(j+1)}\,\mathrm{Re}\Bigg(Y_{j\,1}(\Omega)\, e^{-i\varphi} \Bigg)$$
$$+ \sum_{m=1}^{j} \sqrt{j(j+1) - m(m+1)} \Bigg(\mathrm{Re}\Bigg(S_{jm}(r) Y_{j\,(m+1)}(\Omega)\, e^{-i\varphi} \Bigg)$$
$$+ \mathrm{Re}\Bigg(r \frac{\partial}{\partial r} S_{jm}(r) Y_{j\,(m+1)}(\Omega)\, e^{-i\varphi} \Bigg) \Bigg)$$
$$- \sqrt{j(j+1) - m(m-1)} \Bigg(\mathrm{Re}\Bigg(S_{jm}(r) Y_{j\,(m-1)}(\Omega)\, e^{i\varphi} \Bigg)$$
$$+ \mathrm{Re}\Bigg(r \frac{\partial}{\partial r} S_{jm}(r) Y_{j\,(m-1)}(\Omega)\, e^{i\varphi} \Bigg) \Bigg) \Bigg]. \tag{B.35}$$

The φ-component of $\boldsymbol{B}^{\mathsf{P}}$ is given by eq. (B.28) and applying the product rule leads to

$$B^{\mathsf{P}}_{\varphi} = \frac{1}{r \sin\vartheta} \sum_{jm} \Bigg(S_{jm}(r) + r \frac{\partial}{\partial r} S_{jm}(r) \Bigg) \frac{\partial}{\partial \varphi} Y_{jm}(\Omega),$$

which can be simplified with eq. (A.19)

$$B^{\mathsf{P}}_{\varphi} = \frac{i}{r \sin\vartheta} \sum_{jm} m \Bigg(S_{jm}(r) + r \frac{\partial}{\partial r} S_{jm}(r) \Bigg) Y_{jm}(\Omega).$$

Due to the requirement $m \geq 0$ for the order of the SHR, we reformulate the last equation with the substitution $-m = \nu$ and we find

$$B^{\mathsf{P}}_{\varphi} = \frac{i}{r \sin\vartheta} \Bigg[\sum_{j=1}^{j_{\max}} \sum_{m=1}^{j} m \Bigg(S_{jm}(r) + r \frac{\partial}{\partial r} S_{jm}(r) \Bigg) Y_{jm}(\Omega)$$
$$- \sum_{\nu=1}^{j} \nu \Bigg(S_{j\,-\nu}(r) + r \frac{\partial}{\partial r} S_{j\,-\nu}(r) \Bigg) Y_{j\,-\nu}(\Omega) \Bigg].$$

With the relations for complex conjugate SH and coefficients in eq. (A.7) and (A.41), we derive the following expression:

$$B^{\mathsf{P}}_{\varphi} = \frac{i}{r \sin\vartheta} \Bigg[\sum_{j=1}^{j_{\max}} \sum_{m=1}^{j} m \Bigg(S_{jm}(r) + r \frac{\partial}{\partial r} S_{jm}(r)) Y_{jm}(\Omega)$$
$$- \sum_{\nu=1}^{j} \nu \Bigg(S^{*}_{j\,\nu}(r) + r \frac{\partial}{\partial r} S^{*}_{j\,\nu}(r) \Bigg) Y^{*}_{j\,\nu}(\Omega) \Bigg].$$

Substituting now $\nu = m$ and considering the relation

$$i\big(ab - a^{*}b^{*}\big) = -2\,\mathrm{Im}\,(ab), \tag{B.36}$$

which holds for all complex numbers $a, b \in \mathbb{C}$, leads to the final expression

$$B^{\mathsf{P}}_{\varphi} = \frac{-2}{r \sin\vartheta} \sum_{j=1}^{j_{\max}} \sum_{m=1}^{j} m\ \mathrm{Im}\Bigg(\Bigg(S_{jm}(r) + r \frac{\partial}{\partial r} S_{jm}(r) \Bigg) Y_{jm}(\Omega) \Bigg). \tag{B.37}$$

Analogous to the derivation of the poloidal components, we have to formulate related expressions for the toroidal components. In this case, only the ϑ- and φ-component exist and for the first we find by applying eq. (A.19) on eq. (B.30)

$$B^{\mathsf{T}}_{\vartheta} = \frac{i}{\sin\vartheta} \sum_{jm} m\, T_{jm}(r) Y_{jm}(\Omega).$$

To fulfill the restriction $m \geq 0$, we can follow the derivation of B_φ^{P} above, which leads here to

$$B_\vartheta^{\mathsf{T}} = \frac{-2}{\sin\vartheta} \sum_{j=1}^{j_{\max}} \sum_{m=1}^{j} m \,\mathrm{Im}\Big(T_{jm}(r) Y_{jm}(\Omega)\Big). \tag{B.38}$$

For the last missing component B_φ^{T}, eq. (B.31) leads to

$$B_\varphi^{\mathsf{T}} = -\sum_{jm} T_{jm}(r) \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega).$$

The partial derivative can be resolved by eq. (A.20) and the equation reads:

$$\begin{aligned} B_\varphi^{\mathsf{T}} = \frac{-1}{2} \sum_{jm} T_{jm}(r) \Big[& \sqrt{j(j+1) - m(m+1)} Y_{j\,(m+1)}(\Omega) e^{-i\varphi} \\ & - \sqrt{j(j+1) - m(m-1)} Y_{j\,(m-1)}(\Omega) e^{i\varphi} \Big]. \end{aligned}$$

We also face here the restriction, that $m \geq 0$ has to be satisfied. Therefore, we apply the splitting of the second summation for $m > 0$, $m = 0$ and $m < 0$, and following the derivation of B_ϑ^{P} we obtain

$$\begin{aligned} B_\varphi^{\mathsf{T}} = -\sum_{j=1}^{j_{\max}} \Bigg[& T_{j0}(r) \sqrt{j(j+1)} \,\mathrm{Re}\big(Y_{j1}(\Omega) e^{-i\varphi}\big) \\ & + \sum_{m=1}^{j} \Bigg(\sqrt{j(j+1) - m(m+1)} \,\mathrm{Re}\big(T_{jm}(r) Y_{j\,(m+1)}(\Omega) e^{-i\varphi}\big) \\ & - \sqrt{j(j+1) - m(m-1)} \,\mathrm{Re}\big(T_{jm}(r) Y_{j\,(m-1)}(\Omega) e^{i\varphi}\big) \Bigg) \Bigg], \end{aligned} \tag{B.39}$$

where also eq. (B.34) is used.

Spherical-harmonic representation of the EM torque

C

C.1 Additional derivations for the axial toroidal EM torque

In eq. (2.54), the summation over the index m is not restricted to $m \geq 0$ as assumed in the relation between the Gauss coefficients and the complex field-generating scalar S. Here, we summarize the derivation of eq. (2.55) and we split eq. (2.54) as follows:

$$
\begin{aligned}
L_z^{\mathrm{T}} = \frac{r^2}{\mu_0} \sum_{j=1}^{j_{\max}} j(j+1) \Bigg[& \sum_{m=0}^{j} (j-1) \sqrt{\frac{(j-m-1)^2 - m^2}{(2j+1)(2j-1)}} S_{jm}(r) T^*_{(j-1)\,m}(r) \\
& - (j-2) \sqrt{\frac{(j+1)^2 - m^2}{(2j+3)(2j+1)}} S_{jm}(r) T^*_{(j+1)\,m}(r) \\
& + (j-1) \sqrt{\frac{(j-1)^2}{(2j+1)(2j-1)}} S_{j0}(r) T^*_{(j-1)\,0}(r) - (j+2) \sqrt{\frac{(j+1)^2}{(2j+3)(2j+1)}} S_{j0}(r) T^*_{(j+1)\,0}(r) \\
& + \sum_{m=-j}^{-1} (j-1) \sqrt{\frac{(j-m-1)^2 - m^2}{(2j+1)(2j-1)}} S_{jm}(r) T^*_{(j-1)\,m}(r) \\
& - (j-2) \sqrt{\frac{(j+1)^2 - m^2}{(2j+3)(2j+1)}} S_{jm}(r) T^*_{(j+1)\,m}(r) \Bigg].
\end{aligned}
$$

We use the substitution $m = -\nu$ and the relation for the complex conjugate coefficients in eq. (A.41) to derive

$$S_{j\,-\nu}(r) = (-1)^{-\nu} S^*_{j\,\nu}(r), \tag{C.1}$$

$$T^*_{j\,-\nu}(r) = (-1)^{-\nu} T_{j\,\nu}(r). \tag{C.2}$$

Applying this relation to the last summation leads to

$$
\begin{aligned}
& \sum_{\nu=1}^{j} (j-1) \sqrt{\frac{(j+\nu-1)^2 - \nu^2}{(2j+1)(2j-1)}} S_{j\,-\nu}(r) T^*_{(j-1)\,-\nu}(r) \\
& - (j-2) \sqrt{\frac{(j+1)^2 - \nu^2}{(2j+3)(2j+1)}} S_{j\,-\nu}(r) T^*_{(j+1)\,-\nu}(r) \\
& = \sum_{\nu=1}^{j} (j-1) \sqrt{\frac{(j+\nu-1)^2 - \nu^2}{(2j+1)(2j-1)}} S^*_{j\,\nu}(r) T_{(j-1)\,\nu}(r) \\
& - (j-2) \sqrt{\frac{(j+1)^2 - \nu^2}{(2j+3)(2j+1)}} S^*_{j\,\nu}(r) T_{(j+1)\,\nu}(r),
\end{aligned}
$$

which reads with the re-substitution $\nu = m$ as follows:

$$
\begin{aligned}
& = \sum_{m=1}^{j} (j-1) \sqrt{\frac{(j+m-1)^2 - m^2}{(2j+1)(2j-1)}} S^*_{jm}(r) T_{(j-1)\,m}(r) \\
& - (j-2) \sqrt{\frac{(j+1)^2 - m^2}{(2j+3)(2j+1)}} S^*_{jm}(r) T_{(j+1)\,m}(r).
\end{aligned} \tag{C.3}
$$

With this findings, we can derive eq. (2.55) straightforward.

C.2 Additional derivations for non-axial poloidal EM torques

Here, we want to derive eq. (2.71) from eq. (2.70). In the first step, we split the summation over m into $m > 0$, $m = 0$ and $m < 0$, which reads as follows:

$$\begin{aligned}\boldsymbol{L}^{\mathrm{P}} = \frac{i\,r}{\mu_0}\Bigg\{\sum_{j=1}^{j_{\max}} j(j+1)\Bigg[\sum_{m=1}^{j}\sqrt{j(j+1)-m(m+1)}S_{jm}\Bigg(S^*_{j\,(m+1)}(r)\\ + r\frac{\partial}{\partial r}S^*_{j\,(m+1)}(r)\Bigg) + \sqrt{j(j+1)}S_{j0}(r)\Bigg(S^*_{j1}(r) + r\frac{\partial}{\partial r}S^*_{j1}(r)\Bigg)\\ + \sum_{m=-j}^{-1}\sqrt{j(j+1)-m(m+1)}S_{jm}\Bigg(S^*_{j\,(m+1)}(r) + r\frac{\partial}{\partial r}S^*_{j\,(m+1)}(r)\Bigg)\Bigg]\Bigg\}.\end{aligned} \tag{C.4}$$

We substitute now $m = -\nu$ in the last summation and use the relation eqs. (C.1)–(C.2), which leads to

$$\begin{aligned}&\sum_{\nu=1}^{j}\sqrt{j(j+1)-\nu(\nu-1)}S_{j\,-\nu}(r)\Bigg(S^*_{j\,-(\nu-1)}(r) + r\frac{\partial}{\partial r}S^*_{j\,-(\nu+1)}(r)\Bigg)\\ &= \sum_{\nu=1}^{j}\sqrt{j(j+1)-\nu(\nu-1)}(-1)^{-\nu}S^*_{j\,\nu}(r)\Bigg((-1)^{-(\nu-1)}S_{j\,(\nu-1)}(r)\\ &+ r(-1)^{-(\nu-1)}\frac{\partial}{\partial r}S_{j\,(\nu-1)}(r)\Bigg).\end{aligned}$$

This can be transformed with the re-substitution $\nu = m$ to:

$$= -\sum_{m=1}^{j}\sqrt{j(j+1)-m(m-1)}S^*_{jm}(r)\Bigg(S_{j\,(m-1)}(r) + r\frac{\partial}{\partial r}S_{j\,(m-1)}(r)\Bigg). \tag{C.5}$$

With this last expression and eq. (C.4), we can now derive eq. (2.71).

C.3 Additional derivations for non-axial toroidal EM torque

To perform the integration over φ in eq. (2.75) we apply the following expression,

$$\int_0^{2\pi} e^{i\,n\varphi}d\varphi = \begin{cases} 2\pi & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}, \tag{C.6}$$

which leads to a Kronecker symbol for the related indices.

Moreover, we need the relation for K in eq. (2.77) to perform the integration over ϑ. For the derivation of this relation, we follow Kautzleben (1965). He gives a few recursion formulae for Legendre functions, from which we start here (Kautzleben, 1965, eqs. 256 a, 259 a & 266):

$$P_{k\,(m+2)}(\cos\vartheta) - 2(m+1)\cot\vartheta P_{k\,(m+1)}(\cos\vartheta) + (k+m)(k+m+1)P_{km}(\cos\vartheta) = 0, \tag{C.7}$$

$$\cos\vartheta P_{km}(\cos\vartheta) = \frac{1}{2k+1}\big((k-m+1)P_{(k+1)\,m}(\cos\vartheta) + (k+m)P_{(k-1)\,m}(\cos\vartheta)\big), \tag{C.8}$$

$$\begin{aligned}&-(2k+1)\sin\vartheta P_{k\,(m+1)}(\cos\vartheta)\\ &= (k-m)(k-m+1)P_{(k+1)\,m}(\cos\vartheta) - (k+m)(k+m+1)P_{(k-1)\,m}(\cos\vartheta).\end{aligned} \tag{C.9}$$

Applying the first relation, eq. (C.7), and considering of the definition of the cot-function yields

$$\begin{aligned}&\cos\vartheta\sin\vartheta\frac{\partial}{\partial\vartheta}P_{k\,(m+1)}(\cos\vartheta) + (m+1)P_{k\,(m+1)}(\cos\vartheta)\\ &= (m+1)\big(1-\cos^2\vartheta\big)\cdot P_{k\,(m+1)}(\cos\vartheta) + (k-m)(k+m+1)\cos\vartheta\sin\vartheta P_{km}(\cos\vartheta),\\ &= \sin\vartheta\Big[(m+1)\sin\vartheta P_{k\,(m+1)}(\cos\vartheta) + (k-m)(k+m+1)\cos\vartheta P_{km}(\cos\vartheta)\Big].\end{aligned}$$

We apply here eqs. (C.9) and (C.8) and find:

$$
\begin{aligned}
&= \frac{\sin\vartheta}{2k+1}\Big[(m+1)\big(-(k+m)(k-m+1)P_{(k+1)\,m}(\cos\vartheta) + (k+m)(k+m+1)P_{(k-1)\,m}(\cos\vartheta)\big)\Big] \\
&+ (k-m)(k+m+1)\big((k-m+1)P_{(k+1)\,m}(\cos\vartheta) + (k+m)P_{(k-1)\,m}(\cos\vartheta)\big)\Big], \\
&= \frac{\sin\vartheta}{2k+1}\Big[(k-m+1)(k-m)(-m-1+k+m+1)P_{(k+1)\,m}(\cos\vartheta) \\
&+ (k+m+1)(k+m)(m+1+k-m)P_{(k-1)\,m}(\cos\vartheta)\Big].
\end{aligned}
$$

This can be simplified to

$$
\begin{aligned}
&\cos\vartheta\sin\vartheta\frac{\partial}{\partial\vartheta}P_{k\,(m+1)}(\cos\vartheta) + (m+1)P_{k\,(m+1)}(\cos\vartheta) \\
&= \frac{\sin\vartheta}{2k+1}\Big[(k-m+1)(k-m)kP_{(k+1)\,m}(\cos\vartheta) + (k+m+1)(k+m)(k+1)P_{(k-1)\,m}(\cos\vartheta)\Big],
\end{aligned}
$$

which is the relation for K in eq. (2.77).

The last step to derive eq. (2.79) from eq. (2.78) is the restriction of the summation over m to positive values. We split the summation over m into $m > 0$, $m = 0$ and $m < 0$:

$$
\begin{aligned}
\boldsymbol{L}^{\mathsf{T}} = -\frac{r^2}{\mu_0}\sum_{j=1}^{j_{\max}} j(j+1)\Bigg[&\sum_{m=1}^{j} S_{jm}(r)\Bigg(\frac{(j-m)(j-m-1)(j-1)}{(2j-1)}\,T^*_{(j-1)\,(m+1)}(r) \\
&+ \frac{(j+m+2)(j+m+1)(j+2)}{(2j+3)}\,T^*_{(j+1)\,(m+1)}(r)\Bigg) \\
&+ S_{j0}(r)\Bigg(\frac{j(j-1)^2}{(2j-1)}\,T^*_{(j-1)\,1}(r) + \frac{(j+1)(j+2)^2}{(2j+3)}\,T^*_{(j+1)\,1}(r)\Bigg) \\
&+ \sum_{m=-j}^{-1} S_{jm}(r)\Bigg(\frac{(j-m)(j-m-1)(j-1)}{(2j-1)}\,T^*_{(j-1)\,(m+1)}(r) \\
&+ \frac{(j+m+2)(j+m+1)(j+2)}{(2j+3)}\,T^*_{(j+1)\,(m+1)}(r)\Bigg)\Bigg].
\end{aligned}
\tag{C.10}
$$

We substitute now $m = -\nu$ and use eqs. (C.1) and (C.2) to reformulate the last summation, which leads to

$$
\begin{aligned}
&\sum_{\nu=1}^{j} S_{j\,-\nu}(r)\Bigg(\frac{(j+\nu)(j+\nu-1)(j-1)}{(2j+1)}\,T^*_{(j-1)\,-(\nu-1)}(r) \\
&\quad+ \frac{(j-\nu+2)(j-\nu+1)(j+2)}{(2j+3)}\,T^*_{(j+1)\,-(\nu-1)}(r)\Bigg) \\
&= \sum_{\nu=1}^{j}(-1)^{\nu}S^*_{j\nu}(r)\Bigg(\frac{(j+\nu)(j+\nu-1)(j-1)}{(2j+1)}\,(-1)^{(\nu-1)}\,T_{(j-1)\,(\nu-1)}(r) \\
&\quad+ \frac{(j-\nu+2)(j-\nu+1)(j+2)}{(2j+3)}\,(-1)^{(\nu-1)}\,T_{(j+1)\,(\nu-1)}(r)\Bigg).
\end{aligned}
$$

This can be transformed with the re-substitution $\nu = m$ to:

$$
\begin{aligned}
= -\sum_{m=1}^{j} S^*_{jm}(r)\Bigg(&\frac{(j+m)(j+m-1)(j-1)}{(2j-1)}\,T_{(j-1)\,(m-1)}(r) \\
&+ \frac{(j-m+2)(j-m+1)(j+2)}{(2j+3)}\,T_{(j+1)\,(m-1)}(r)\Bigg).
\end{aligned}
\tag{C.11}
$$

With the last expression and eq. (C.10) we can derive eq. (2.79).

Additional derivation for the toroidal magnetic field D

D.1 Derivation of the scalar induction equations

Here, we derive the scalar form of the induction equation for the mantle (eq. (3.9)-(3.10)) and the outer-core (eq. (3.16)-(3.17)) domain. Due to the fact that the mantle equations are a simplified special case of the outer core equations (reduction of the V, V^{e}, U and U^{e} related terms), we derive first the more general case of the induction equation for the outer-core domain.

To derive the scalar toroidal induction equation for the outer-core domain, we split eq. (3.8) into toroidal and poloidal parts. The toroidal part is given by:

$$\frac{1}{\mu_0}\operatorname{rot}\left(\frac{1}{\sigma_{\mathsf{C}}}\operatorname{rot}\boldsymbol{B}^{\mathsf{T}}\right)-\operatorname{rot}\left(\left(\boldsymbol{u}\times\boldsymbol{B}\right)^{\mathsf{P}}+\boldsymbol{E}^{\mathsf{eP}}\right)+\frac{\partial}{\partial t}\boldsymbol{B}^{\mathsf{T}}=0. \tag{D.1}$$

Here, we have considered that for any poloidal vector $\boldsymbol{U}^{\mathsf{P}}$ follows: $\operatorname{rot}\boldsymbol{U}^{\mathsf{P}}$ is toroidal and vice versa and $a\boldsymbol{B}^{\mathsf{P}}$ is poloidal if a is a scalar function of r only. Now, we apply the definitions in eqs. (2.25) and (3.11)–(3.14) to the toroidal equation above.

$$\frac{1}{\mu_0}\operatorname{rot}\left(\frac{1}{\sigma_{\mathsf{C}}}\operatorname{rot}\operatorname{rot}(\boldsymbol{T})\right)-\operatorname{rot}\left(\boldsymbol{r}V+\operatorname{grad}W+\boldsymbol{r}V^{\mathsf{e}}+\operatorname{grad}W^{\mathsf{e}}\right)+\frac{\partial}{\partial t}\operatorname{rot}(\boldsymbol{r}T)=0$$

With the following relations for the differential operator rot, any vector $\boldsymbol{v}$ and any scalar s (e. g. Bronstein et al., 1997, Sec. 13.2.5),

$$\operatorname{rot}(s\boldsymbol{v})=s\operatorname{rot}\boldsymbol{v}+\operatorname{grad}s\times\boldsymbol{v}, \tag{D.2}$$

$$\operatorname{rot}(\boldsymbol{v}_1+\boldsymbol{v}_2)=\operatorname{rot}\boldsymbol{v}_1+\operatorname{rot}\boldsymbol{v}_2, \tag{D.3}$$

$$\operatorname{rot}(\operatorname{grad}s)=0, \tag{D.4}$$

the toroidal induction equation reads:

$$\frac{1}{\mu_0}\left[\frac{1}{\sigma_{\mathsf{C}}}\operatorname{rot}\operatorname{rot}\operatorname{rot}(\boldsymbol{r}T)+\operatorname{grad}\frac{1}{\sigma_{\mathsf{C}}}\times\operatorname{rot}\operatorname{rot}(\boldsymbol{r}T)\right]-\operatorname{rot}(\boldsymbol{r}V)-\operatorname{rot}(\boldsymbol{r}V^{\mathsf{e}})+\frac{\partial}{\partial t}\operatorname{rot}(\boldsymbol{r}T)=0.$$

In addition to the relations in eqs. (2.28)-(2.29), we consider (e. g. Krause & Rädler, 1980, Sec. 13.3)

$$\operatorname{rot}\operatorname{rot}\operatorname{rot}(\boldsymbol{r}T)=\boldsymbol{r}\times\operatorname{grad}(\Delta T), \tag{D.5}$$

which leads to the following toroidal induction equation:

$$\frac{1}{\mu_0\sigma_{\mathsf{C}}}\left[\boldsymbol{r}\times\operatorname{grad}(\Delta T)\right]+\frac{1}{\mu_0}\left[\operatorname{grad}\frac{1}{\sigma_{\mathsf{C}}}\times\left(-\boldsymbol{r}\Delta T+\operatorname{grad}\left(\frac{\partial}{\partial r}rT\right)\right)\right]$$
$$+\boldsymbol{r}\times\operatorname{grad}V+\boldsymbol{r}\times\operatorname{grad}V^{\mathsf{e}}-\frac{\partial}{\partial t}(\boldsymbol{r}\times\operatorname{grad}T)=0. \tag{D.6}$$

For the further derivation, we have to reformulate the second term in the equation above, which reads with $\boldsymbol{r}=r\boldsymbol{e}_r$, a spherical symmetric conductivity σ and its related partial derivative $\boldsymbol{e}_r\frac{\partial}{\partial r}\frac{1}{\sigma}=-\boldsymbol{r}\frac{1}{r\sigma^2}\frac{\partial}{\partial r}\sigma$:

$$\frac{1}{\mu_0}\left[\left(\boldsymbol{r}\frac{1}{r\sigma_{\mathsf{C}}^2}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\times\boldsymbol{r}\Delta T\right)-\left(\boldsymbol{r}\frac{1}{r\sigma_{\mathsf{C}}^2}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\times\operatorname{grad}\left(\frac{\partial}{\partial r}rT\right)\right)\right]=$$
$$\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\left[-\boldsymbol{r}\times\operatorname{grad}\left(\frac{1}{r\sigma_{\mathsf{C}}}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\frac{\partial}{\partial r}rT\right)\right]. \tag{D.7}$$

We also consider that due to the sole r-dependence of the conductivity the following relation is valid:

$$\boldsymbol{r}\frac{1}{r\sigma_{\mathsf{C}}}\frac{\partial}{\partial r}\sigma_{\mathsf{C}} \times \operatorname{grad}\left(\frac{\partial}{\partial r} rT\right) = \boldsymbol{r} \times \operatorname{grad}\left(\frac{1}{r\sigma_{\mathsf{C}}}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\frac{\partial}{\partial r} rT\right).$$

First, we exchange the second term in eq. (D.6) by eq. (D.7) and, secondly, we consider that the following is valid for any scalar function a and b:

$$\boldsymbol{r} \times \operatorname{grad} a + \boldsymbol{r} \times \operatorname{grad} b = \boldsymbol{r} \times \operatorname{grad}(a+b).$$

This leads for the toroidal induction equation of the core domain to:

$$\boldsymbol{r} \times \operatorname{grad}\left(\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\left[\Delta T - \frac{1}{r\sigma_{\mathsf{C}}}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\frac{\partial}{\partial r}(rT) + V + V^{\mathsf{e}} - \frac{\partial}{\partial t}T\right]\right) = 0. \tag{D.8}$$

In a simplified notation, eq. (D.8) reads

$$\boldsymbol{r} \times \operatorname{grad} f(r,\Omega) = 0\,.$$

Performing the vector product shows the sole r-dependence of the expression, which leads to $f(r,\Omega) = g(r)$. Based on the normalization of the field-generating scalars in eqs. (2.27) and (3.15), we can conclude that

$$\int_\Omega \frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\left[\Delta T - \frac{1}{r\sigma_{\mathsf{C}}}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\frac{\partial}{\partial r}(rT) + V + V^{\mathsf{e}} - \frac{\partial}{\partial t}T\right]\mathrm{d}\Omega = 0\,.$$

If $\int_\Omega T\,\mathrm{d}\Omega = 0$ like in eq. (2.27) then it also holds that $\int_\Omega \Delta T\,\mathrm{d}\Omega = 0$, seen when considering the splitting of the Laplace operator according to eq. (A.16). From $\boldsymbol{r} \times \operatorname{grad}(f(r,\Omega)) = 0$, then it follows in this line of argumentation that $f(r,\Omega) = g(r)$ vanishes identically, i.e. eq. (D.8) is only satisfied if

$$\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\left(\Delta T - \frac{1}{r\sigma_{\mathsf{C}}}\frac{\partial}{\partial r}\sigma_{\mathsf{C}}\frac{\partial}{\partial r}(rT) + V + V^{\mathsf{e}} - \frac{\partial}{\partial t}T\right) = 0\,. \tag{D.9}$$

This leads to eq. (3.16) in sec. 3.1. If we neglect the additional term in the induction equation of the core and exchange the conductivity of the core, σ_{C}, by the r-dependent conductivity profile of the mantle, $\sigma_{\mathsf{M}}(r)$, we can analogously find eq. (3.9).

We split the induction equation for the core given in eq. (3.8) into its poloidal and toroidal parts, where the poloidal part is given by:

$$\frac{1}{\mu_0}\operatorname{rot}\left(\frac{1}{\sigma_{\mathsf{C}}}\operatorname{rot}\boldsymbol{B}^{\mathsf{P}}\right) - \operatorname{rot}\left((\boldsymbol{u}\times\boldsymbol{B})^{\mathsf{T}} + \boldsymbol{E}^{\mathsf{eT}}\right) + \frac{\partial}{\partial t}\boldsymbol{B}^{\mathsf{P}} = 0. \tag{D.10}$$

Here, we consider in addition the definitions of the field-generating scalars in eqs. (2.25) and (3.11)–(3.14) and find

$$\frac{1}{\mu_0}\operatorname{rot}\left(\frac{1}{\sigma_{\mathsf{C}}}\operatorname{rot}\operatorname{rot}\operatorname{rot}(\boldsymbol{r}S)\right) - \operatorname{rot}\left(\operatorname{rot}(\boldsymbol{r}U) + \operatorname{rot}(\boldsymbol{r}U^{\mathsf{e}})\right) + \frac{\partial}{\partial t}\operatorname{rot}\operatorname{rot}(\boldsymbol{r}S) = 0.$$

With the relation in eq. (D.3) and reversing the order of partial and temporal derivatives, the induction equation reads:

$$\operatorname{rot}\left[\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\left(\operatorname{rot}\operatorname{rot}\operatorname{rot}(\boldsymbol{r}S)\right) - \operatorname{rot}(\boldsymbol{r}U) - \operatorname{rot}(\boldsymbol{r}U^{\mathsf{e}}) + \operatorname{rot}\left(\boldsymbol{r}\frac{\partial}{\partial t}S\right)\right] = 0.$$

For the further derivation, we apply eqs. (D.5) and (2.29), which leads to

$$\operatorname{rot}\left[\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}(\boldsymbol{r}\times\operatorname{grad}\Delta S) + \boldsymbol{r}\times\operatorname{grad}U + \boldsymbol{r}\times\operatorname{grad}U^{\mathsf{e}} - \boldsymbol{r}\times\operatorname{grad}\left(\frac{\partial}{\partial t}S\right)\right] = 0.$$

As discussed above, it is possible to move the sole r-dependent factor $\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}$ into the grad operator, and we reformulate the poloidal induction as follows:

$$\operatorname{rot}\left[\boldsymbol{r}\times\operatorname{grad}\left(\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\Delta S + U + U^{\mathsf{e}} - \frac{\partial}{\partial t}S\right)\right] = 0. \tag{D.11}$$

With the relation in eq. (2.29), we find in simplified notation for the equation above

$$\mathrm{rot}\big[\boldsymbol{r}\times\mathrm{grad}\, f(r,\Omega)\big] = -\,\mathrm{rot}\big(\mathrm{rot}\,\boldsymbol{r} f(r,\Omega)\big) = 0$$

The term $(\mathrm{rot}\,\boldsymbol{r} f(r,\Omega))$ is sole toroidal and in Krause & Rädler (1980, Sec. 13.4) a relation for a toroidal field $\boldsymbol{A}^{\mathsf{T}}$ is given, which reads: if $\mathrm{rot}\,\boldsymbol{A}^{\mathsf{T}} = 0$ in the whole region then $\boldsymbol{A}^{\mathsf{T}} = 0$. Therefore, it holds

$$\mathrm{rot}\,\boldsymbol{r} f(r,\Omega) = 0 = \boldsymbol{r}\times\mathrm{grad}\, f(r,\Omega)\,,$$

which reduces the problem to that solved above for the toroidal field. With the same arguments like for the toroidal part, we can conclude that the equation above is only valid, if

$$\frac{1}{\mu_0\,\sigma_{\mathsf{C}}}\Delta S + U + U^{\mathsf{e}} - \frac{\partial}{\partial t}S = 0, \tag{D.12}$$

is fulfilled, taking into account the normalization of the field-generating scalars in eqs. (2.26) and (3.15), and its consequences for Δ.

D.2 SHR of the scalar induction equations

For the further derivation of the toroidal magnetic field and for the basis equation of the NHDC, we need the spherical harmonic representation (SHR) of the scalar induction equations (3.9) and (3.10) for the mantle. First, we focus on the toroidal induction equation and apply the SHR of the field-generating scalar T in eq. (2.52):

$$\frac{1}{\mu_0\sigma_{\mathsf{M}}(r)}\left[\sum_{jm}\Delta T_{jm}(r,t)Y_{jm}(\Omega) - \frac{1}{r\sigma_{\mathsf{M}}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{\mathsf{M}}(r)\sum_{jm}\frac{\partial}{\partial r}\big(rT_{jm}(r,t)\big)Y_{jm}(\Omega)\right] - \sum_{jm}\frac{\partial}{\partial t}T_{jm}(r,t)Y_{jm}(\Omega) = 0.$$

We split now the Laplace operator into its radial and angular part, according to eqs. (A.16) and (A.17). Applying this relations and resorting the different terms in one summation leads to

$$\sum_{jm}\left\{\frac{1}{\mu_0\sigma_{\mathsf{M}}(r)}\left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\big(r^2\frac{\partial}{\partial r}T_{jm}(r,t)\big) - j(j+1)T_{jm}(r,t)\right)\right.\right.$$
$$\left.\left. -\frac{1}{r\sigma_{\mathsf{M}}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{\mathsf{M}}(r)\frac{\partial}{\partial r}\big(rT_{jm}(r,t)\big)\right] - \frac{\partial}{\partial t}T_{jm}(r,t)\right\}Y_{jm}(\Omega) = 0. \tag{D.13}$$

Due to the orthogonality of the SH, we can conclude that each equation for any j and m has to be satisfied. Implementing the radial derivatives we can find the equation:

$$\frac{1}{\mu_0\sigma_{\mathsf{M}}(r)}\left[\frac{2}{r}\frac{\partial}{\partial r}T_{jm}(r,t) + \frac{\partial^2}{\partial r^2}T_{jm}(r,t) - \frac{j(j+1)}{r^2}T_{jm}(r,t)\right.$$
$$\left. -\frac{1}{\sigma_{\mathsf{M}}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{\mathsf{M}}(r)\left(T_{jm}(r,t) + r\frac{\partial}{\partial r}T_{jm}(r,t)\right)\right] - \frac{\partial}{\partial t}T_{jm}(r,t) = 0, \tag{D.14}$$

where a reordering with respect to the order of the partial derivatives of the coefficients T_{jm} leads to eq. (3.18).

In the second part of this section, we derive the SHR of the poloidal induction equation in eq. (3.10) for the mantle, applying eq. (2.43):

$$\frac{1}{\mu_0\sigma_{\mathsf{M}}(r)}\sum_{jm}\Delta S_{jm}(r,t)Y_{jm}(\Omega) - \sum_{jm}\frac{\partial}{\partial t}S_{jm}(r,t)Y_{jm}(\Omega) = 0,$$

which can be further simplified using eq. (A.16) and considering eq. (A.18)

$$\sum_{jm}\left\{\frac{1}{\mu_0\sigma_{\mathsf{M}}(r)}\left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\big(r^2\frac{\partial}{\partial r}S_{jm}(r,t)\big) - j(j+1)S_{jm}(r,t)\right)\right] - \frac{\partial}{\partial t}S_{jm}(r,t)\right\}Y_{jm}(\Omega) = 0.$$

Due to the orthogonality of the SH, we can conclude that each equation for any j and m has to be satisfied. This leads to:

$$\frac{\partial^2}{\partial r^2}S_{jm}(r,t)+\frac{2}{r}\frac{\partial}{\partial r}S_{jm}(r,t)-\frac{j(j+1)}{r^2}S_{jm}(r,t)-\mu_0\sigma_{\mathrm{M}}(r)\frac{\partial}{\partial t}S_{jm}(r,t)=0, \tag{D.15}$$

i. e. eq. (3.19).

D.3 SHR of the field-generating scalar W

Beside the equivalence of eq. (3.12) and (3.50), we present in this section the detailed derivation of the SHR of the field generating scalar W.

First, we consider the general definition

$$(\boldsymbol{u}\times\boldsymbol{B})=\boldsymbol{r}V+\operatorname{grad}W-\boldsymbol{r}\times\operatorname{grad}U, \tag{D.16}$$

following from eqs. (3.11) and (3.12). Then we put $\boldsymbol{u}\times\boldsymbol{B}$ into the right-hand side of eq. (3.50) and obtain

$$\begin{aligned}\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]&=\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{r}V+\operatorname{grad}W-\boldsymbol{r}\times\operatorname{grad}U\big)\big],\\&=\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\boldsymbol{r}V+\boldsymbol{r}\times\operatorname{grad}W-\boldsymbol{r}\big(\boldsymbol{r}\cdot\operatorname{grad}U\big)+\big(\boldsymbol{r}\cdot\boldsymbol{r}\big)\operatorname{grad}U\big],\end{aligned}$$

where the first term vanishes identically. In the last two terms, the $\boldsymbol{e}_r$-parts cancel each other and the remaining term cancels by multiplication of rot-operator with $\boldsymbol{r}$. With eq. (2.29) follows

$$\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]=-\boldsymbol{r}\operatorname{rot}\operatorname{rot}\big(\boldsymbol{r}W\big),$$

which can be rewritten considering eq. (2.28)

$$\begin{aligned}\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]&=-\boldsymbol{r}\left[-\boldsymbol{r}\Delta W+\operatorname{grad}\left(\frac{\partial}{\partial r}rW\right)\right],\\&=r^2\Delta W-\boldsymbol{r}\operatorname{grad}\left(\frac{\partial}{\partial r}rW\right).\end{aligned}$$

According to the definition (A.16) and $\boldsymbol{r}=r\boldsymbol{e}_r$, it holds

$$\begin{aligned}\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]&=\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}W\right)+\Delta_\Omega W-r\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}rW\right),\\&=\Delta_\Omega W\,,\end{aligned} \tag{D.17}$$

showing the equivalence of the expressions in eqs. (3.12) and (3.50).

For the derivation of eq. (3.51), we apply the angular Laplace operator on the SHR of W, according to eq. (A.18),

$$\begin{aligned}\Delta_\Omega W(r,\Omega,t)&=\sum_{jm}W_{jm}(r,t)\,\Delta_\Omega Y_{jm}(\Omega),\\&=-\sum_{jm}j(j+1)W_{jm}(r,t)Y_{jm}(\Omega).\end{aligned}$$

Next, we divide this equation by $-j(j+1)$ and multiply it with $Y^*_{jm}(\Omega)$. Considering the orthogonality condition in eq. (A.6), leads then to

$$W_{jm}=\frac{-1}{j(j+1)}\int\limits_\Omega\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,.$$

For the further derivation, we reformulate the integral kernel by use of $u_r=0$ at $r=R_{\mathrm{CMB}}$, obtaining

$$\boldsymbol{r}\cdot\operatorname{rot}\big[\boldsymbol{r}\times\big(\boldsymbol{u}\times\boldsymbol{B}\big)\big]=\boldsymbol{r}\cdot\operatorname{rot}\big(\boldsymbol{u}\,r\,B_r\big),$$

which explicitly reads

$$\big[\boldsymbol{r}\cdot\mathrm{rot}\big(\boldsymbol{u}\,r\,B_r\big)\big]_r = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\big(r\,B_r u_\varphi\sin\vartheta\big) - \frac{\partial}{\partial\varphi}\big(r\,B_r u_\vartheta\big)\right].$$

For the next step, we express the vector components on the right-hand side by the related field-generating scalars, this are for the magnetic field

$$B_r = \frac{-1}{r}\,\Delta_\Omega\,S, \tag{D.18}$$

as defined in eq. (2.31), and the new definitions for the velocity field

$$u_\vartheta = \frac{\partial}{\partial\vartheta}P + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Q, \tag{D.19}$$

$$u_\varphi = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}P - \frac{\partial}{\partial\vartheta}Q, \tag{D.20}$$

which are similar to eqs. (B.23) and (B.24) for the magnetic field ($P\hat{=}\frac{1}{r}\frac{\partial}{\partial r}(rS)$, $Q\hat{=}T$). The use of this definitions leads to the following equation

$$W_{jm} = \frac{-1}{j(j+1)}\int\limits_\Omega \frac{1}{\sin\vartheta}\left\{\frac{\partial}{\partial\vartheta}\left[-\Delta_\Omega\,S\left(\frac{\partial}{\partial\varphi}P - \sin\vartheta\frac{\partial}{\partial\vartheta}Q\right)\right]\right.$$
$$\left. - \frac{\partial}{\partial\varphi}\left[-\Delta_\Omega\,S\left(\frac{\partial}{\partial\vartheta}P + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Q\right)\right]\right\}Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,. \tag{D.21}$$

Now, we apply the product rule for partial derivatives obtaining:

$$W_{jm} = \frac{-1}{j(j+1)}\int\limits_\Omega \frac{1}{\sin\vartheta}\left[-\frac{\partial}{\partial\vartheta}\Delta_\Omega\,S\left(\frac{\partial}{\partial\varphi}P - \sin\vartheta\frac{\partial}{\partial\vartheta}Q\right)\right.$$
$$- \Delta_\Omega\,S\left(\frac{\partial^2}{\partial\vartheta\partial\varphi}P - \frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}Q\right) + \frac{\partial}{\partial\varphi}\Delta_\Omega\,S\left(\frac{\partial}{\partial\vartheta}P + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Q\right)$$
$$\left. + \Delta_\Omega\,S\left(\frac{\partial^2}{\partial\varphi\partial\vartheta}P + \frac{1}{\sin\vartheta}\frac{\partial^2}{\partial\varphi^2}Q\right)\right]\,Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,. \tag{D.22}$$

For the field-generating scalars, we choose the following SHR:

$$S(r,\Omega,t) = \sum_{kl} S_{kl}(r,t)Y_{kl}(\Omega), \tag{D.23}$$

$$P(r,\Omega,t) = \sum_{st} P_{st}(r,t)Y_{st}(\Omega), \tag{D.24}$$

$$Q(r,\Omega,t) = \sum_{st} Q_{st}(r,t)Y_{st}(\Omega). \tag{D.25}$$

With eq. (A.18), we find

$$\Delta_\Omega\,S(r,\Omega,t) = \sum_{kl} -k(k+1)S_{kl}(r,t)Y_{kl}(\Omega). \tag{D.26}$$

In the further derivation, we use for simplification the notation without any arguments for the SHR of the field-generating scalars. With this representation, eq. (D.22) reads

$$
\begin{aligned}
W_{jm} = \frac{-1}{j(j+1)} \int\limits_{\Omega} \frac{1}{\sin\vartheta} \Bigg\{ & \frac{\partial}{\partial\vartheta}\Bigg(\sum_{kl} k(k+1) S_{kl} Y_{kl}(\Omega)\Bigg)\Bigg(\frac{\partial}{\partial\varphi}\sum_{st} P_{st}Y_{st}(\Omega) - \sin\vartheta\frac{\partial}{\partial\vartheta}\sum_{st} Q_{st}Y_{st}(\Omega)\Bigg) \\
& + \sum_{kl} k(k+1)S_{kl}Y_{kl}(\Omega)\Bigg[\frac{\partial^2}{\partial\vartheta\partial\varphi}\sum_{st} P_{st}Y_{st}(\Omega) - \frac{\partial}{\partial\vartheta}\Bigg(\sin\vartheta\frac{\partial}{\partial\vartheta}\sum_{st} Q_{st}Y_{st}(\Omega)\Bigg)\Bigg] \\
& - \frac{\partial}{\partial\vartheta}\Bigg(\sum_{kl} k(k+1)S_{kl}Y_{kl}(\Omega)\Bigg)\Bigg(\frac{\partial}{\partial\vartheta}\sum_{st} P_{st}Y_{st}(\Omega) + \frac{1}{\sin\vartheta}\sum_{st} Q_{st}Y_{st}(\Omega)\Bigg) \\
& - \sum_{kl} k(k+1)S_{kl}Y_{kl}(\Omega)\Bigg(\frac{\partial^2}{\partial\varphi\partial\vartheta}\sum_{st} P_{st}Y_{st}(\Omega) + \frac{1}{\sin\vartheta}\frac{\partial^2}{\partial\varphi^2}\sum_{st} Q_{st}Y_{st}(\Omega)\Bigg)\Bigg\} Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,. \qquad \text{(D.27)}
\end{aligned}
$$

After introducing the abbreviation

$$
\sum_{klst} = \sum_{k=1}^{k_{\text{max}}}\sum_{l=-k}^{k}\sum_{s=1}^{s_{\text{max}}}\sum_{t=-s}^{s}, \qquad \text{(D.28)}
$$

and resorting eq. (D.27) with respect to the field-generating scalars, we obtain

$$
\begin{aligned}
W_{jm} = \frac{-1}{j(j+1)} \int\limits_{\Omega} \frac{1}{\sin\vartheta}\sum_{klst} k(k+1)S_{kl}\Bigg\{ & P_{st}\Bigg[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) + Y_{kl}(\Omega)\frac{\partial^2}{\partial\vartheta\partial\varphi}Y_{st}(\Omega) \\
& - \frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega) - Y_{kl}(\Omega)\frac{\partial^2}{\partial\vartheta\partial\varphi}Y_{st}(\Omega)\Bigg] \\
& + Q_{st}\Bigg[-\Bigg(\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\Bigg)\sin\vartheta\Bigg(\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\Bigg) - Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}\Bigg(\sin\vartheta\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\Bigg) \\
& - \frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{st}(\Omega) - Y_{kl}(\Omega)\frac{1}{\sin\vartheta}\frac{\partial^2}{\partial\varphi^2}Y_{st}(\Omega)\Bigg]\Bigg\} Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,, \\
W_{jm} = \frac{-1}{j(j+1)} \int\limits_{\Omega} \sum_{klst} k(k+1)S_{kl}\Bigg\{ & \frac{1}{\sin\vartheta}P_{st}\Bigg[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) - \frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\Bigg] \\
& - Q_{st}\Bigg[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) + \frac{1}{\sin^2\vartheta}Y_{kl}(\Omega)\frac{\partial^2}{\partial\varphi^2}Y_{st}(\Omega) \\
& + \frac{1}{\sin\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}\Bigg(\sin\vartheta\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\Bigg)\Bigg]\Bigg\} Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,. \qquad \text{(D.29)}
\end{aligned}
$$

We use the relation,

$$
Y_{kl}(\Omega)\,\Delta_\Omega\,Y_{st}(\Omega) = Y_{kl}(\Omega)\Bigg(\frac{1}{\sin\vartheta}\frac{\partial^2}{\partial\varphi^2}Y_{st}(\Omega) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\Bigg(\sin\vartheta\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\Bigg)\Bigg)\,, \qquad \text{(D.30)}
$$

to define the following coupling integrals,

$$
\begin{aligned}
\mathbf{K}^{jm}_{klst} &= \int\limits_{\Omega}\Bigg[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) \\
&\qquad + Y_{kl}(\Omega)\,\Delta_\Omega\,Y_{st}(\Omega)\Bigg]Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,. \qquad \text{(D.31)} \\
\mathbf{L}^{jm}_{klst} &= \int\limits_{\Omega}\frac{1}{\sin\vartheta}\Bigg[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) - \frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\Bigg]Y^*_{jm}(\Omega)\,\mathrm{d}\Omega\,, \qquad \text{(D.32)}
\end{aligned}
$$

The calculation of the SHR of the field-generating scalar W is then described by

$$
W_{jm} = \frac{-1}{j(j+1)}\sum_{klst} k(k+1)S_{kl}\big[\mathbf{L}^{jm}_{klst}\,P_{st} - \mathbf{K}^{jm}_{klst}\,Q_{st}\big]\,, \qquad \text{(D.33)}
$$

which is equivalent with eq. (3.52).

D.4 Derivation of the coupling integrals $\mathbf{K}_{klst}^{jm}$ and $\mathbf{L}_{klst}^{jm}$

In section 3.3, a relation between the field-generating scalars W_{jm}, S_{kl}, P_{st} and Q_{st} using the coupling integrals $\mathbf{K}_{klst}^{jm}$ and $\mathbf{L}_{klst}^{jm}$ is given by eq. (3.52). Here, we present the derivation of the eqs. (3.53) and (3.54), based on the expression (D.31) and (D.32) in appendix D.3.

The coupling integral in eq. (D.31) is reformulated, considering the relation (A.18), as follows

$$\mathbf{K}_{klst}^{jm} = \int\limits_{\Omega} \left[\frac{\partial}{\partial\vartheta} Y_{kl}(\Omega) \frac{\partial}{\partial\vartheta} Y_{st}(\Omega) + \frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} Y_{kl}(\Omega) \frac{\partial}{\partial\varphi} Y_{st}(\Omega) \right] Y_{jm}^*(\Omega)\,\mathrm{d}\Omega \\ - s(s+1) \int\limits_{\Omega} Y_{kl}(\Omega)\,Y_{st}(\Omega)\,Y_{jm}^*(\Omega)\,\mathrm{d}\Omega\,. \tag{D.34}$$

For the last integral, we find in Varshalovich et al. (1989, Sec. 5.9.1, eq. 4)

$$\int\limits_{\Omega} Y_{kl}(\Omega) Y_{st}(\Omega) Y_{jm}^*(\Omega)\,\mathrm{d}\Omega = \sqrt{\frac{(2k+1)(2s+1}{4\pi(2j+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{j\,0}\, \mathbf{C}_{k\,l\,s\,t}^{j\,m}\,. \tag{D.35}$$

The relation

$$\frac{\partial}{\partial\vartheta} Y_{kl}(\Omega) \frac{\partial}{\partial\vartheta} Y_{st}(\Omega) + \frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} Y_{kl}(\Omega) \frac{\partial}{\partial\varphi} Y_{st}(\Omega) = \frac{1}{2} \sum_{ru} \left[k(k+1) + s(s+1) - r(r+1) \right] \mathbf{Q}_{klst}^{ru}\, Y_{ru}(\Omega), \tag{D.36}$$

is given in Pěč & Martinec (1988, eq. 11) with

$$\mathbf{Q}_{klst}^{ru} = \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2r+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{r\,0}\, \mathbf{C}_{k\,l\,s\,t}^{r\,u}\,. \tag{D.37}$$

Considering the expression (D.36) by the reformulation of the first integral in eq. (D.34) leads to

$$\int\limits_{\Omega} \left[\frac{\partial}{\partial\vartheta} Y_{kl}(\Omega) \frac{\partial}{\partial\vartheta} Y_{st}(\Omega) + \frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} Y_{kl}(\Omega) \frac{\partial}{\partial\varphi} Y_{st}(\Omega) \right] Y_{jm}^*(\Omega)\,\mathrm{d}\Omega \\ = \int\limits_{\Omega} \left[\frac{1}{2} \sum_{ru} \left[k(k+1) + s(s+1) - r(r+1) \right] \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2r+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{r\,0}\, \mathbf{C}_{k\,l\,s\,t}^{r\,u}\, Y_{ru}(\Omega) \right] Y_{jm}^*(\Omega)\,\mathrm{d}\Omega, \\ = \frac{1}{2} \left[k(k+1) + s(s+1) - j(j+1) \right] \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2j+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{j\,0}\, \mathbf{C}_{k\,l\,s\,t}^{j\,m}. \tag{D.38}$$

Next, we combine eqs. (D.35) and (D.38) to determine the first coupling integral

$$\mathbf{K}_{klst}^{jm} = \frac{1}{2} \left[k(k+1) + s(s+1) - j(j+1) \right] \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2j+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{j\,0}\, \mathbf{C}_{k\,l\,s\,t}^{j\,m} \\ - s(s+1) \sqrt{\frac{(2k+1)(2s+1}{4\pi(2j+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{j\,0}\, \mathbf{C}_{k\,l\,s\,t}^{j\,m}\,,$$

which can be simplified to

$$\mathbf{K}_{klst}^{jm} = \frac{1}{2} \left[k(k+1) - s(s+1) - j(j+1) \right] \sqrt{\frac{(2k+1)(2s+1)}{4\pi(2j+1)}}\, \mathbf{C}_{k\,0\,s\,0}^{j\,0}\, \mathbf{C}_{k\,l\,s\,t}^{j\,m}.$$

In this way, we have determined the first coupling integral, given in eq. (3.53).

In the following, we summarize the derivation for the second coupling integral, given in eq. (3.54), which is more extensive than for the first coupling integral. Therefore, we split this derivation into the basic steps in this appendix, while we present the derivation of necessary relations in the appendix D.5.

We start from eq. (D.32),

$$\mathsf{L}_{klst}^{jm} = \int\limits_{\Omega} \frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) - \frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\right]Y_{jm}^{*}(\Omega)\,\mathrm{d}\Omega\,,$$

which can be reformulated, as shown in appendix D.5, to

$$\mathsf{L}_{klst}^{jm} = \int\limits_{\Omega} \boldsymbol{e}_r \cdot \left[\boldsymbol{S}_{kl}^{(0)}(\Omega) \times \boldsymbol{S}_{st}^{(0)}(\Omega)\right] Y_{jm}^{*}(\Omega)\,\mathrm{d}\Omega\,. \tag{D.39}$$

There, we use the definition of the vector spherical harmonics in eq. (A.24). With the following relation

$$\boldsymbol{S}_{jm}^{(0)}(\Omega) = i\,\sqrt{j(j+1)}\,\mathbb{Y}_{jm}^{j}(\Omega), \tag{D.40}$$

which is derived in eq. (D.45), later on we can write

$$\mathsf{L}_{klst}^{jm} = -\int\limits_{\Omega} \sqrt{k(k+1)s(s+1)}\,\boldsymbol{e}_r \cdot \left[\mathbb{Y}_{kl}^{k}(\Omega) \times \mathbb{Y}_{st}^{s}(\Omega)\right] Y_{jm}^{*}(\Omega)\,\mathrm{d}\Omega\,. \tag{D.41}$$

Here, $\mathbb{Y}_{jm}^{j}(\Omega)$ denotes the vector spherical harmonics, as defined in Varshalovich et al. (1989, Sec. 7.3.1, eqs. 6 and 9). In appendix D.5, a relation is given for the integral kernel in terms of Clebsch-Gordan coefficients. The application of eq. (D.58) on (D.41) leads to

$$\begin{aligned}\mathsf{L}_{klst}^{jm} = \int\limits_{\Omega} \frac{i}{2}\frac{1}{\sqrt{4\pi}}(2k+1)(2s+1)&\left[\sum_{ru} \mathsf{C}_{k\,0\,s\,0}^{r+1\,0}\,\mathsf{C}_{k\,l\,s\,t}^{r\,u}\,Y_{ru}(\Omega)\right.\\ &\left.\sqrt{\frac{(k+s+r+2)(k+s-r)(k-s+r+1)(-k+s+r+1)}{k(k+1)(2k+1)\,s(s+1)(2s+1)(2r+3)}}\right] Y_{jm}^{*}(\Omega)\,\mathrm{d}\Omega\,.\end{aligned} \tag{D.42}$$

Considering the orthogonality condition (A.6) and reducing the equation, we end up with

$$\begin{aligned}\mathsf{L}_{klst}^{jm} = \frac{i}{2}&\sqrt{\frac{(2k+1)(2s+1)}{4\pi(2j+3)}}\,\mathsf{C}_{k\,0\,s\,0}^{j+1\,0}\,\mathsf{C}_{k\,l\,s\,t}^{j\,m}\\ &\sqrt{(k+s+j+2)(k+s-j)(k-s+j+1)(-k+s+j+1)}\,,\end{aligned}$$

which is equivalent with eq. (3.54).

D.5 Additional relation for the derivation of L_{klst}^{jm}

As explained in the section above, we summarize here the extensive derivation for different relations of vector spherical harmonics, which are needed for the determination of L_{klst}^{jm}. First, we show the equivalence of eqs. (D.31) and (D.39), starting with

$$\boldsymbol{S}_{kl}^{(0)}(\Omega) \times \boldsymbol{S}_{st}^{(0)}(\Omega) = \left[\boldsymbol{e}_\varphi\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega) - \boldsymbol{e}_\vartheta\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\right] \times \left[\boldsymbol{e}_\varphi\frac{\partial}{\partial\vartheta}Y_{st}(\Omega) - \boldsymbol{e}_\vartheta\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{st}(\Omega)\right]$$

where only the definition (A.24) is used. Considering the vector products between the unit vectors $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_\vartheta$ leads to

$$\boldsymbol{S}_{kl}^{(0)}(\Omega) \times \boldsymbol{S}_{st}^{(0)}(\Omega) = \boldsymbol{e}_r\,\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}Y_{kl}(\Omega)\frac{\partial}{\partial\varphi}Y_{st}(\Omega) - \frac{\partial}{\partial\varphi}Y_{kl}(\Omega)\frac{\partial}{\partial\vartheta}Y_{st}(\Omega)\right]. \tag{D.43}$$

Hereby, the equivalence of the mentioned equation is shown.

Furthermore, expression (D.40) is based on the definition of vector spherical harmonics in Varshalovich et al. (1989, Sec. 7.3.1, eqs. 6 and 9)

$$\mathbb{Y}^{j}_{jm}(\Omega) = \frac{-i}{\sqrt{j(j+1)}}\,(\boldsymbol{e}_r \times \nabla_\Omega) Y_{jm}(\Omega)\,. \tag{D.44}$$

With the definition in eq. (A.24), this leads to

$$\boldsymbol{S}^{(0)}_{jm}(\Omega) = i\,\sqrt{j(j+1)}\,\mathbb{Y}^{j}_{jm}(\Omega). \tag{D.45}$$

To reduce the integral kernel in eq. (D.41), we use the relation given in Varshalovich et al. (1989, Sec. 7.3.10, eq. 101)

$$\mathbb{Y}^{k}_{kl}(\Omega) \times \mathbb{Y}^{s}_{st}(\Omega) = i\,\sqrt{\frac{3}{2\pi}}(2k+1)(2s+1)\sum_{rup}\begin{Bmatrix} k & s & r \\ k & s & p \\ 1 & 1 & 1 \end{Bmatrix}\mathbf{C}^{t\,0}_{k\,0\,s\,0}\,\mathbf{C}^{r\,u}_{k\,l\,s\,t}\,\mathbb{Y}^{p}_{ru}(\Omega)\,, \tag{D.46}$$

where the coefficient matrix denotes the Wigner $9j$ symbols as defined in Varshalovich et al. (1989, Sec. 10.1). We need only the r-component of this expression, which is given by

$$\begin{aligned}\boldsymbol{e}_r \cdot \left[\mathbb{Y}^{k}_{kl}(\Omega) \times \mathbb{Y}^{s}_{st}(\Omega)\right] = i\,\sqrt{\frac{3}{2\pi}}(2k+1)(2s+1)\sum_{ru}\mathbf{C}^{r\,u}_{k\,l\,s\,t}\left[\begin{Bmatrix} k & s & r \\ k & s & r-1 \\ 1 & 1 & 1 \end{Bmatrix}\mathbf{C}^{r-1\,0}_{k\,0\,s\,0}\frac{r}{\sqrt{r(2r+1)}}Y_{ru}(\Omega)\right.\\ \left. - \begin{Bmatrix} k & s & r \\ k & s & r+1 \\ 1 & 1 & 1 \end{Bmatrix}\mathbf{C}^{r+1\,0}_{k\,0\,s\,0}\frac{(r+1)}{\sqrt{(r+1)(2r+1)}}Y_{ru}(\Omega)\right]\,,\end{aligned} \tag{D.47}$$

where the two remaining values for the summation over p, $p = r-1$ and $p = r+1$ are applied (by the definition of $\mathbb{Y}^{l}_{jm}(\Omega)$). In addition, we need expressions for $\boldsymbol{e}_r \cdot \mathbb{Y}^{r-1}_{ru}(\Omega)$ and $\boldsymbol{e}_r \cdot \mathbb{Y}^{r+1}_{ru}(\Omega)$. With the relations from Varshalovich et al. (1989, Sec. 7.3.1, eqs. 6, 7 and 10) we find

$$\begin{aligned}\boldsymbol{e}_r \cdot \mathbb{Y}^{r-1}_{ru}(\Omega) &= \boldsymbol{e}_r\left[\sqrt{\frac{r+1}{2r+1}}\frac{1}{\sqrt{r(r+1)}}\nabla_\Omega\,Y_{ru}(\Omega) + \sqrt{\frac{r}{2r+1}}\,\boldsymbol{e}_r\,Y_{ru}(\Omega)\right],\\ &= \frac{r}{\sqrt{r(2r+1)}}\,Y_{ru}(\Omega)\,,\end{aligned} \tag{D.48}$$

and

$$\begin{aligned}\boldsymbol{e}_r \cdot \mathbb{Y}^{r+1}_{ru}(\Omega) &= \boldsymbol{e}_r\left[\sqrt{\frac{r}{2r+1}}\frac{1}{\sqrt{r(r+1)}}\nabla_\Omega\,Y_{ru}(\Omega) - \sqrt{\frac{r+1}{2r+1}}\,\boldsymbol{e}_r\,Y_{ru}(\Omega)\right],\\ &= -\frac{(r+1)}{\sqrt{(r+1)(2r+1)}}\,Y_{ru}(\Omega).\end{aligned} \tag{D.49}$$

Here, we have considered the scalar product of $\boldsymbol{e}_r$ with itself and with the angular nabla operator (A.15). The most extensive derivation to find the required relation is now the reformulation of the summation of Clebsch-Gordan coefficients. In a further step of the derivation, we have to determine the Winger $9j$ symbols by tabulated values. With the relations

$$\mathbf{C}^{c\,0}_{a\,0\,b+2\,0} = (-1)^a\sqrt{\frac{2c+1}{2b+5}}\,\mathbf{C}^{b+2\,0}_{a\,0\,b\,0}\,, \tag{D.50}$$

$$\mathbf{C}^{c\,0}_{a\,0\,b\,0} = (-1)^a\sqrt{\frac{2c+1}{2b+1}}\,\mathbf{C}^{b\,0}_{a\,0\,c\,0}\,, \tag{D.51}$$

taken from Varshalovich et al. (1989, Sec. 8.4.3, eq. 10), we find

$$\mathbf{C}^{b+2\,0}_{a\,0\,c\,0} = (-1)^a\sqrt{\frac{2b+5}{2c+1}}\,\mathbf{C}^{c\,0}_{a\,0\,b+2\,0}\,, \tag{D.52}$$

which transforms according to eq. (23) of Varshalovich et al. (1989, Sec. 8.6.4) and the relations above to

$$\mathbf{C}_{a\,0\,c\,0}^{b+2\,0} = -\sqrt{\frac{2b+5}{2c+1}}\,\mathbf{C}_{a\,0\,c\,0}^{b\,0}\sqrt{\frac{(a+b+c+2)(a+b-c+1)(a-b+c)(-a+b+c+1)}{(a+b+c+3)(a+b-c+2)(a-b+c-1)(-a+b+c+2)}}\,. \tag{D.53}$$

To reformulate eq. (D.47), we choose now $a = k$, $b = r-1$ and $c = s$, which leads to

$$\mathbf{C}_{k\,0\,s\,0}^{r-1\,0} = -\sqrt{\frac{2r-1}{2r+3}}\,\mathbf{C}_{k\,0\,s\,0}^{r+1\,0}\sqrt{\frac{(k+s+r+2)(k-s+r+1)(k+s-r)(-k+s+r+1)}{(k+s+r+1)(k-s+r)(k+s-r+1)(-k+s+r)}}\,. \tag{D.54}$$

Now, we can express both terms in the summation in eq. (D.47) with the same Clebsch-Gordan coefficient and find

$$\begin{aligned} \boldsymbol{e}_r \cdot \left[\mathbb{Y}_{kl}^k(\Omega) \times \mathbb{Y}_{st}^s(\Omega)\right] = i\sqrt{\frac{3}{2\pi}}(2k+1)(2s+1) \sum_{ru} \mathbf{C}_{k\,l\,s\,t}^{r\,u}\,\mathbf{C}_{k\,0\,s\,0}^{r+1\,0}\,Y_{ru}(\Omega) \Bigg[& \begin{Bmatrix} k & s & r \\ k & s & r-1 \\ 1 & 1 & 1 \end{Bmatrix} \frac{-r}{\sqrt{r(2r+1)}} \\ & \sqrt{\frac{2r-1}{2r+3}}\sqrt{\frac{(k+s+r+2)(k-s+r+1)(k+s-r)(-k+s+r+1)}{(k+s+r+1)(k-s+r)(k+s-r+1)(-k+s+r)}} \\ & - \begin{Bmatrix} k & s & r \\ k & s & r+1 \\ 1 & 1 & 1 \end{Bmatrix} \frac{(r+1)}{\sqrt{(r+1)(2r+1)}} \Bigg]\,. \end{aligned} \tag{D.55}$$

The Wigner $9j$-symbols in the equation above have to be determined for the further derivation. According to Varshalovich et al. (1989, Tab. 10.9) for $\lambda = \mu = 0$ and $\nu = 1$ (which are used in the cited table) is the first Wigner $9j$-symbol given by

$$\begin{Bmatrix} k & s & r \\ k & s & r-1 \\ 1 & 1 & 1 \end{Bmatrix} = \frac{r}{2}\sqrt{\frac{(k+s+r+1)(k+s-r+1)(k-s+r)(-k+s+r)}{6k(k+1)(2k+1)s(s+1)(2s+1)r(2r-1)(2r+1)}}\,. \tag{D.56}$$

The second Wigner $9j$-symbol is for $\lambda = \mu = 0$ and $\nu = -1$ given by

$$\begin{Bmatrix} k & s & r \\ k & s & r+1 \\ 1 & 1 & 1 \end{Bmatrix} = \frac{r+1}{2}\sqrt{\frac{(k+s+r+2)(k+s-r)(k-s+r+1)(-k+s+r+1)}{6k(k+1)(2k+1)s(s+1)(2s+1)(r+1)(2r+1)(2r+3)}}\,. \tag{D.57}$$

Here, we have also reduced the expressions as much as possible. With both last expressions for the Wigner $9j$-symbols we can rewrite eq. (D.55), which then reads

$$\begin{aligned} \boldsymbol{e}_r \cdot \left[\mathbb{Y}_{kl}^k(\Omega) \times \mathbb{Y}_{st}^s(\Omega)\right] = -\frac{i}{2}\,\frac{(2k+1)(2s+1)}{\sqrt{4\pi}} & \sum_{ru} \mathbf{C}_{k\,0\,s\,0}^{r+1\,0}\,\mathbf{C}_{k\,l\,s\,t}^{r\,u}\,Y_{ru}(\Omega) \\ & \sqrt{\frac{(k+s+r+2)(k+s-r)(k-s+r+1)(-k+s+r+1)}{k(k+1)(2k+1)s(s+1)(2s+1)(2r+3)}}\,. \end{aligned} \tag{D.58}$$

We have now everything prepared to derive $\mathbf{L}_{klst}^{jm}$ from eq. (D.41), where we express the integral kernel by eq. (D.58):

$$\begin{aligned} \mathbf{L}_{klst}^{jm} = \frac{i}{2}\,\frac{(2k+1)(2s+1)}{\sqrt{4\pi}} & \Bigg[\sum_{ru} \mathbf{C}_{k\,0\,s\,0}^{r+1\,0}\,\mathbf{C}_{k\,l\,s\,t}^{r\,u} \\ & \sqrt{\frac{(k+s+r+2)(k+s-r)(k-s+r+1)(-k+s+r+1)}{k(k+1)(2k+1)\,s(s+1)(2s+1)(2r+3)}}\;\underbrace{\int\limits_{\Omega} Y_{ru}(\Omega)Y_{jm}^*(\Omega)\,\mathrm{d}\Omega}_{=\delta_{rj}\delta_{um}}\Bigg]\,. \end{aligned} \tag{D.59}$$

With orthonormal condition (A.6), indicated in the equation above, we find for the indices $r = j$ and $u = m$

$$\begin{aligned} \mathbf{L}_{klst}^{jm} = \frac{i}{2} & \sqrt{\frac{(2k+1)\,(2s+1)}{4\pi\,(2j+3)}}\,\mathbf{C}_{k\,0\,s\,0}^{j+1\,0}\,\mathbf{C}_{k\,l\,s\,t}^{j\,m} \\ & \sqrt{(k+s+j+2)(k+s-j)(k-s+j+1)(-k+s+j+1)}\,, \end{aligned} \tag{D.60}$$

which is identical with the expression shown in eq. (3.54) in section 3.3.

D.6 Matrix notation of the finite-difference approach

The matrix notation of the set of equations describing the discrete BVP in sec. (3.4.1) uses the definition of **T** by eq. (3.66). We have thereby also defined the related vector of the right-hand sides, **R**. The tri-diagonal matrix $\mathbb{A}$, constructed by the coefficients of eqs. (3.61), (3.62) and (3.65), has the dimension of $(i_{\text{max}} + 1) \times (i_{\text{max}} + 1)$, and is given by

$$\mathbb{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 & 0 & 0 & \dots & & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} & 0 & 0 & \dots & & 0 \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} & \mathrm{A}_{34} & 0 & \dots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathrm{A}_{\mathrm{i_{max}\,i_{max}}} & \mathrm{A}_{\mathrm{i_{max}\,i_{max}+1}} \\ 0 & 0 & 0 & 0 & 0 & \dots & \mathrm{A}_{\mathrm{i_{max}+1\,i_{max}}} & \mathrm{A}_{\mathrm{i_{max}+1\,i_{max}+1}} \end{bmatrix}, \tag{D.61}$$

where the non-zero elements are

$$\begin{aligned} \mathrm{A}_{11} &= -\Big[\frac{2}{(\Delta r)^2} + \frac{1}{R_{\text{CMB}}}\big(\Phi_0 - \frac{2}{\Delta r}\big) + \Theta_0\Big]\,, \\ \mathrm{A}_{12} &= \Big[\frac{2}{(\Delta r)^2} + \frac{1}{\Delta r}\Phi_0\Big]\,, \\ \mathrm{A}_{\mathrm{i,i-1}} &= \Big[\frac{1}{(\Delta r)^2} + \frac{1}{2\Delta r}\Phi_{i-1}\Big]\,, \\ \mathrm{A}_{\mathrm{ii}} &= -\Big[\frac{2}{(\Delta r)^2} + \Theta_{i-1}\Big]\,, \\ \mathrm{A}_{\mathrm{i\,i+1}} &= \Big[\frac{1}{(\Delta r)^2} - \frac{1}{2\Delta r}\Phi_{i-1}\Big]\,, \\ \mathrm{A}_{\mathrm{i_{max}+1\,i_{max}+1}} &= 1\,. \end{aligned}$$

The vector of right-hand sides is given by

$$\mathbf{R} = \left[\mathrm{R}_0, \dots, \mathrm{R}_{i_{\text{max}}}\right]^\top \tag{D.62}$$

where

$$\mathrm{R}_0 = \frac{\mu_0 \sigma_{\text{M}} W_{jm}}{R_{\text{CMB}}}\left(\Phi_0 - \frac{2}{\Delta r}\right), \tag{D.63}$$

$$\mathrm{R}_i = 0 \qquad \text{for} \quad i \in [1, i_{\text{max}}]. \tag{D.64}$$

D.7 Matrix notation of the Crank-Nicolson approach

Here, we determine the matrix elements of $\mathbb{A}$, given in eq. (3.73). With the following abbreviations,

$$\alpha_i := \frac{\Delta t}{2(\Delta r)^2 \Psi_i}\,, \tag{D.65}$$

$$\beta_i := \frac{\Delta t \Phi_i}{4 \Delta r \Psi_i}\,, \tag{D.66}$$

$$\gamma_i := \frac{\Delta t \Theta_i}{2\Psi_i}\,, \tag{D.67}$$

we can rewrite the discrete partial differential equation (3.70)

$$\begin{aligned} &-(\alpha_i + \beta_i)\mathsf{T}_{i+1}^{n+1} + (1 + 2\alpha_i + \gamma_i)\mathsf{T}_i^{n+1} + (\alpha_i + \beta_i)\mathsf{T}_{i-1}^{n+1} \\ &= (\alpha_i + \beta_i)\mathsf{T}_{i+1}^{n} + (1 - 2\alpha_i - \gamma_i)\mathsf{T}_i^{n} + (\alpha_i - \beta_i)\mathsf{T}_{i-1}^{n}\,. \end{aligned} \tag{D.68}$$

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This is only valid for $i \in [1, i_{\mathsf{max}} - 1]$. For the special cases $i = 0$ and $i = i_{\mathsf{max}}$, we have to consider the boundary condition like in eqs. (3.62) and (3.65) and find in the notation introduced above for $i = i_{\mathsf{max}}$

$$\mathsf{T}^n_{i_{\mathsf{max}}} = 0, \tag{D.69}$$

and for $i = 0$

$$\begin{aligned} &-(\alpha_0+\beta_0)\mathsf{T}_1^{n+1} + (1+2\alpha_0+\gamma_0)\mathsf{T}_0^{n+1} + (\alpha_0+\beta_0)\big[\frac{2\Delta r}{R_{\mathsf{CMB}}}(\mu_0\sigma_{\mathsf{M}}W^{n+1}_{jm}) + \mathsf{T}_1^{n+1}\big] \\ &= (\alpha_0+\beta_0)\mathsf{T}_1^{n} + (1-2\alpha_0-\gamma_0)\mathsf{T}_0^{n} + (\alpha_0+\beta_0)\big[\frac{2\Delta r}{R_{\mathsf{CMB}}}(\mu_0\sigma_{\mathsf{M}}W^{n}_{jm}) + \mathsf{T}_1^{n}\big]\,. \end{aligned} \tag{D.70}$$

Now, we can define the tri-diagonal $\mathbb{A}$ matrix in eq. (3.73), which has the same dimension $((i_{\mathsf{max}}+1) \times (i_{\mathsf{max}}+1))$ and structure as the matrix in eq. (D.61) by its non-zero elements

$$\begin{aligned} \mathrm{A}_{11} &= \big[(1+2\alpha_0+\gamma_0) + (\alpha_0+\beta_0)\frac{2\Delta r}{R_{\mathsf{CMB}}}\big]\,, \\ \mathrm{A}_{\mathrm{i\,i-1}} &= \big[\alpha_{i-1}+\beta_{i-1}\big]\,, \\ \mathrm{A}_{\mathrm{ii}} &= \big[1+2\alpha_{i-1}+\gamma_{i-1}\big]\,, \\ \mathrm{A}_{\mathrm{i\,i+1}} &= -\big[\alpha_{i-1}+\beta_{i-1}\big]\,, \\ \mathrm{A}_{\mathrm{i_{max}+1\,i_{max}+1}} &= 1\,. \end{aligned} \tag{D.71}$$

(D.72)

The time-dependent vector of the right-hand sides is determined by the relation

$$\mathbf{R}^n = \mathbb{B}\mathbf{T}^n + \mathbf{F}^n\,. \tag{D.73}$$

Here, $\mathbb{B}$ is also a tri-diagonal matrix with the dimension $(i_{\mathsf{max}}+1) \times (i_{\mathsf{max}}+1)$. Its non-zero elements are given by

$$\begin{aligned} \mathrm{B}_{11} &= \big[(1-2\alpha_0-\gamma_0) + (\alpha_0-\beta_0)\frac{2\Delta r}{R_{\mathsf{CMB}}}\big]\,, \\ \mathrm{B}_{12} &= \big[2\alpha_0\big]\,, \\ \mathrm{B}_{\mathrm{i\,i-1}} &= \big[\alpha_{i-1}-\beta_{i-1}\big]\,, \\ \mathrm{B}_{\mathrm{ii}} &= \big[1-2\alpha_{i-1}-\gamma_{i-1}\big]\,, \\ \mathrm{B}_{\mathrm{i\,i+1}} &= \big[\alpha_{i-1}+\beta_{i-1}\big]\,, \\ \mathrm{B}_{\mathrm{i_{max}+1\,i_{max}+1}} &= 1\,. \end{aligned} \tag{D.74}$$

The following relation holds for the vector **F**

$$\mathbf{F}^n = \big[\mathsf{F}_0^n, \ldots, \mathsf{F}^n_{i_{\mathsf{max}}}\big]^\top\,, \tag{D.75}$$

with

$$\mathsf{F}_0^n = \frac{2\Delta r}{R_{\mathsf{CMB}}}\big[\alpha_0\mu_0\sigma_{\mathsf{M}}\big(W^n_{jm} - W^{n+1}_{jm}\big) - \beta_0\mu_0\sigma_{\mathsf{M}}\big(W^n_{jm} - W^{n+1}_{jm}\big)\big]\,, \tag{D.76}$$

$$\mathsf{F}_i^n = 0 \qquad \text{for} \quad i \in [1, i_{\mathsf{max}}]\,. \tag{D.77}$$

With this matrices and vector definitions, we can now write the Crank-Nicolson approach in the matrix notation as given by eqs. (3.73) and (3.74).

Relations to magnetic measurements E

E.1 Definition of the Gauss coefficients

Following the classical approach of Gauss (e.g. Jacobs, 1987, Chap. 4) to separate both sources of the geomagnetic potential, ϕ, in a source-free domain between the Earth's surface and the ionosphere, we can assume

$$\phi = \phi^{\mathsf{i}} + \phi^{\mathsf{e}}\,, \tag{E.1}$$

where ϕ^{i} and ϕ^{e} denote the internal and external potentials, respectively. The internal potential is related to the sources in the Earth's interior, whereas the external potential is related to the sources in the ionosphere and magnetosphere. In the Schmidt's normalization, we find for the internal potential:

$$\phi^{\mathsf{i}}(r,\vartheta,\varphi,t) = R_{\mathsf{E}} \sum_{j=1}^{\infty} \left(\frac{R_{\mathsf{E}}}{r}\right)^{j+1} \sum_{m=0}^{j} \left[g_{jm}(t)\cos(m\varphi) + h_{jm}(t)\sin(m\varphi)\right] \hat{P}_{jm}(\cos\vartheta)\,. \tag{E.2}$$

The Legendre function $\hat{P}_{jm}(\cos\vartheta)$ is here used in Schmidt's normalization. By eq. (E.2) are the Gauss coefficients g_{jm} and h_{jm} defined.

E.2 Relation between Gauss coefficients and spherical harmonic representation

A relation between real SH coefficients of S in Ferrers-Neumann normalization and the magnetic measurements by means of Gauss coefficients g_{jm} and h_{jm} is given by Ballani et al. (2002, eq. (21)):

$$S^{\mathsf{c}}_{jm}(r,t) = \frac{1}{j} N_{jm} R_{\mathsf{E}} g_{jm}(t), \tag{E.3}$$

$$S^{\mathsf{s}}_{jm}(r,t) = \frac{1}{j} N_{jm} R_{\mathsf{E}} h_{jm}(t), \tag{E.4}$$

where Schmidt's normalization factor is

$$N_{jm} = \sqrt{(2-\delta_{m0})\frac{(j-m)!}{(j+m)!}}. \tag{E.5}$$

With eq. (A.44), we can derive the following relation between the complex and real SH coefficients analogously to Greiner-Mai et al. (2004)

$$\begin{aligned} S_{jm}(r,t) &= \frac{\lambda_{jm}}{(2-\delta_{m0})}\big(S^{\mathsf{c}}_{jm}(r,t) - i\,S^{\mathsf{s}}_{jm}(r,t)\big), \\ &= \frac{R_{\mathsf{E}}}{j(2-\delta_{m0})} N_{jm}\lambda_{jm}\big(g_{jm}(t) - i\,h_{jm}(t)\big). \end{aligned}$$

With λ_{jm} given by eq. (A.40) follows

$$\begin{aligned} S_{jm}(r,t) &= \frac{(-1)^m R_{\mathsf{E}}}{j(2-\delta_{m0})}\sqrt{(2-\delta_{m0})\frac{(j-m)!}{(j+m)!}\frac{4\pi}{2j+1}\frac{(j+m)!}{(j-m)!}}\big(g_{jm}(t) - i\,h_{jm}(t)\big), \\ &= (-1)^m \frac{R_{\mathsf{E}}}{j}\sqrt{\frac{4\pi}{(2-\delta_{m0})(2j+1)}}\big(g_{jm}(t) - i\,h_{jm}(t)\big). \end{aligned} \tag{E.6}$$

This relation is valid for $j \in [1, j_{\mathsf{max}}]$ and $m \in [0, j]$. For negative m, we use the relation in eq. (A.41).

List of symbols | L

Symbol	Explanation	Page
$\mathbb{A}$	matrix of system of discretized equations of the BVP	23
$\boldsymbol{B}$	magnetic flux vector	3
$\boldsymbol{B}^{\mathsf{P}}$	poloidal magnetic flux	5
$\boldsymbol{B}^{\mathsf{T}}$	toroidal magnetic flux	5
$\mathbb{B}$	matrix of right-hand sides (Crank-Nicolson approach)	64
$\mathbb{C}$	set of complex numbers	8
$\mathbf{C}_{k\,l\,s\,t}^{j\,m}$	Clebsch-Gordan coefficients	22
$\mathrm{d}V$	infinitesimal volume element	4
$\mathrm{d}\Omega$	infinitesimal surface element (in spherical coordinates)	4
$\boldsymbol{e}_i$	unit-base vector	3
$\boldsymbol{E}$	electric field	15
$\boldsymbol{E}_{\mathsf{C}}$	electric field in Earth's outer core	16
$\boldsymbol{E}_{\mathsf{M}}$	electric field in Earth's mantle	16
$\boldsymbol{E}^{\mathsf{e}}$	electric field associated with turbulent flow	16
$\overset{3}{\mathcal{E}}$	Levi-Civita tensor	39
$\boldsymbol{F}$	Lorentz force density	3
F	vector of boundary values (Crank-Nicolson approach)	64
$g_{jm}(t)$	Gauss coefficients of geomagnetic field	65
$h_{jm}(t)$	Gauss coefficients of geomagnetic field	65
$\mathcal{I}$	identity tensor	39
$\boldsymbol{j}$	electric current density	3
$\boldsymbol{j}_{\mathsf{C}}$	electric current density in Earth's outer core	16
$\boldsymbol{j}_{\mathsf{M}}$	electric current density in Earth's mantle	16
$\mathbf{K}_{klst}^{jm}$	coupling coefficients for SHR of field-generating scalar W	22
$\boldsymbol{L}$	electromagnetic core-mantle coupling (EM) torque	3
$\boldsymbol{L}^{\mathsf{P}}$	complex combination of non-axial components of the poloidal EM torque	9
$\boldsymbol{L}^{\mathsf{T}}$	complex combination of non-axial components of the toroidal EM torque	11
$\mathbf{L}_{klst}^{jm}$	coupling coefficients for SHR of field-generating scalar W	22
$\mathcal{M}$	magnetic stress tensor	3
N_{jm}	Schmidt's normalization factor	65
P	field-generating scalar of $\boldsymbol{u}$	57
$P_{jm}(\cos\vartheta)$	associated Legendre function	35
$\tilde{P}_{kl}(\cos\vartheta)$	associated Legendre function in Ferrers-Neumann normalization	35
Q	field-generating scalar of $\boldsymbol{u}$	57
$\mathbf{Q}_{klst}^{ru}$	coupling coefficients for two Clebsch-Gordan coefficients	59
$\boldsymbol{r}$	position vector	4
r	radial component of spherical coordinates	4
R_{CMB}	radius of the core-mantle boundary (CMB)	7
R_{ICB}	radius of the inner-core boundary (ICB)	16
R_σ	outer radius of the conducting part of Earth's mantle	16
R	vector of right-hand sides related to $\mathbb{A}$	23
$\mathbb{R}$	set of real numbers	8
S	field-generating scalar of poloidal magnetic flux	5
$\boldsymbol{S}_{jm}^{(\lambda)}(\Omega)$	vector spherical harmonics	36
T	field-generating scalar of toroidal magnetic flux	5
T	reduced notation of SHR coefficients $T_{jm}(r,t)$	21
$\mathsf{T}_{,r}$	reduced notation of partial derivative of T	21
$\mathsf{T}_{,rr}$	reduced notation of second order partial derivative of T	21

Symbol	Explanation	Page
$\mathsf{T}_{,t}$	reduced notation of time derivative of T	21
$\mathbf{T}$	vector notation of discretized T	23
u_ϑ	component of surface fluid-flow velocity of outer core at R_{CMB}	57
u_φ	component of surface fluid-flow velocity of outer core at R_{CMB}	57
$\boldsymbol{u}$	surface fluid-flow velocity of outer core at R_{CMB}	16
U	field-generating scalar of toroidal part of $\boldsymbol{u} \times \boldsymbol{B}$	17
U^{e}	field-generating scalar of toroidal part of $\boldsymbol{E}^{\text{e}}$	17
V	field-generating scalar of poloidal part of $\boldsymbol{u} \times \boldsymbol{B}$	17
V^{e}	field-generating scalar of poloidal part of $\boldsymbol{E}^{\text{e}}$	17
W	field-generating scalar of poloidal part of $\boldsymbol{u} \times \boldsymbol{B}$	17
W^{e}	field-generating scalar of poloidal part of $\boldsymbol{E}^{\text{e}}$	17
$Y_{jm}(\Omega)$	orthonormal scalar spherical harmonics	35
$\mathbb{Y}^{j}_{jm}(\Omega)$	alternative vector spherical harmonics	61
α_i	abbreviation for elements of $\mathbb{A}$ and $\mathbb{B}$	63
β_i	abbreviation for elements of $\mathbb{A}$ and $\mathbb{B}$	63
γ_i	abbreviation for elements of $\mathbb{A}$ and $\mathbb{B}$	63
δ_{ij}	Kronecker's symbol	3, 35
ϵ_{klm}	Levi-Civita symbol	39
ϑ	angular component of spherical coordinates	4
Θ	abbreviation in partial differential eq. (3.47)	21
λ_{kl}	normalization factor between different Legendre functions	37
μ_0	permeability of the vacuum	3
σ	electric conductivity	15
σ_{C}	electric conductivity of Earth's outer core	16
σ_{M}	electric conductivity of Earth's mantle	16
φ	angular component of spherical coordinates	4
Φ	abbreviation in partial differential eq. (3.47)	21
Ψ	abbreviation in partial differential eq. (3.47)	21
Ω	denotes spherical surface integral $\int_\Omega \ldots \mathrm{d}\Omega$	4
	and abbreviation for angular spherical coordinates (ϑ, φ)	35
Ω_{CMB}	spherical approximation of the core-mantle boundary (CMB)	3
Ω_0	unit sphere	35
∇	nabla operator	35
∇_Ω	angular part of nabla operator	35
Δ	Laplace operator	36
Δ_Ω	angular part of Laplace operator	36

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